Analogue filter calculations for the RTZ shift register DAC

Version 2, Marcel van de Gevel, 2 May 2024

Changes:

Version 2, 2 May 2024: extended with section 6 about combined LC filters/MFB filters

Version 1, January 2024: first version

1. Choice of pole locations

As the DAC is a DSD DAC, there is no possibility to digitally compensate droop of the magnitude response. I've therefore chosen a Butterworth (maximally-flat magnitude) type of filter, fourth-order Butterworth at 80 kHz to be specific. The cut-off frequency is a compromise between the suppression of ultrasonic noise and the bandwidth requirement for feline listeners.

2. Compensating for the effect of finite op-amp gain bandwidth product in an integrator

The circuit of the upper part of figure 1 is a simple op-amp integrator. Assuming that the op-amp is ideal (has nullor properties), its negative input is at ground potential. When the input voltage is V_{in}, a current V_{in}/R flows through the resistor, a voltage $V_{in}/(sRC)$ drops across the capacitor and the output voltage becomes $-V_{in}/(sRC)$, assuming that the resistor and capacitor are also ideal. So far, so good. Conversely, the input voltage is $-sRC\ V_{out}$.

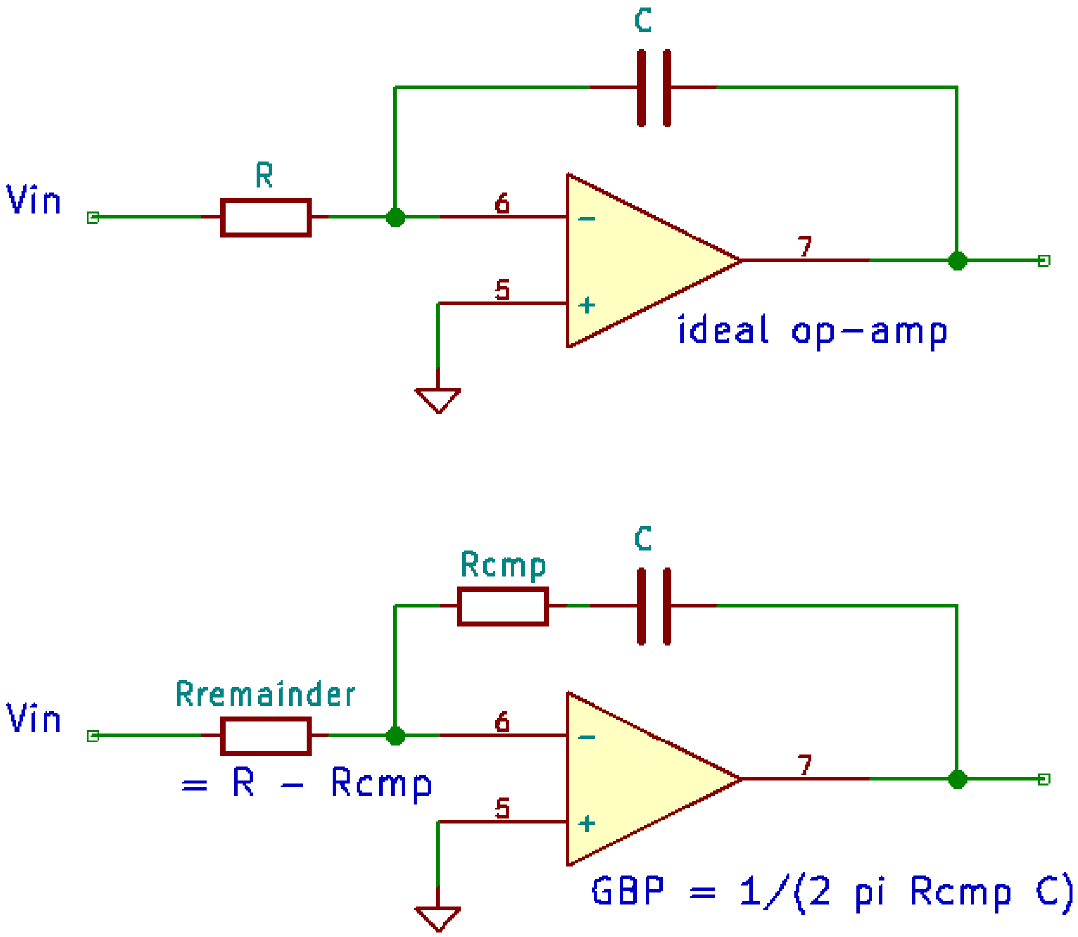

Figure 1: Integrator with an ideal and a non-ideal op-amp

When good resistors and capacitors are used, the main non-ideality in a practical implementation of the circuit is usually the finite gain-bandwidth product of a practical op-amp. With its positive input grounded, an op-amp with open-loop gain $2\pi f_{GBP}/s$ requires an input voltage of $-(s/2\pi f_{GBP})\ V_{out}$ at its negative input to produce an output voltage V_{out}.

Imagine R in the circuit with an ideal op-amp is replaced with a potmeter with the wiper open. Depending on the position of the wiper, the voltage on it can then be anything between $-sRC\ V_{out}$ (wiper turned to the input) and 0 (wiper turned to the virtual ground). As long as $RC \geq 1/2\pi f_{GBP}$, there is a wiper position where the voltage is the $-(s/2\pi f_{GBP})\ V_{out}$ that a non-ideal op-amp would need.

Hence, using an op-amp with finite gain-bandwidth product, the effect of the finite gain-bandwidth product can be compensated for by connecting the inverting input to a tap on the input resistor. Practically, this means that resistor R is split into a part $R_{cmp} = 1/(2\pi f_{GBP}C)$ and $R_{remainder} = R - R_{cmp}$. Resistor R_{cmp} is connected straight in series with the integration capacitor C and the resistor to the negative input is reduced to $R_{remainder}$. This is shown in the bottom circuit of figure 1.

For simplicity, in the rest of this document, the op-amps will be assumed ideal. Most of the circuits can be corrected for finite gain-bandwidth product using the method explained in this section.

3. Replacing integrators with ideal inductors to simplify calculations

Figure 2 shows a subcircuit that's often found in multiple feedback (MFB) filters. When you apply a voltage step from 0 to V at the input, a current $V/R_a + V/R_b$ will immediately start flowing into the input. As the integrator output voltage builds up, the input current increases. All in all, the input impedance is equal to the parallel connection of R_a, R_b and an inductance $L = R_a R_b C_a$, as can be verified by straightforward network analysis.

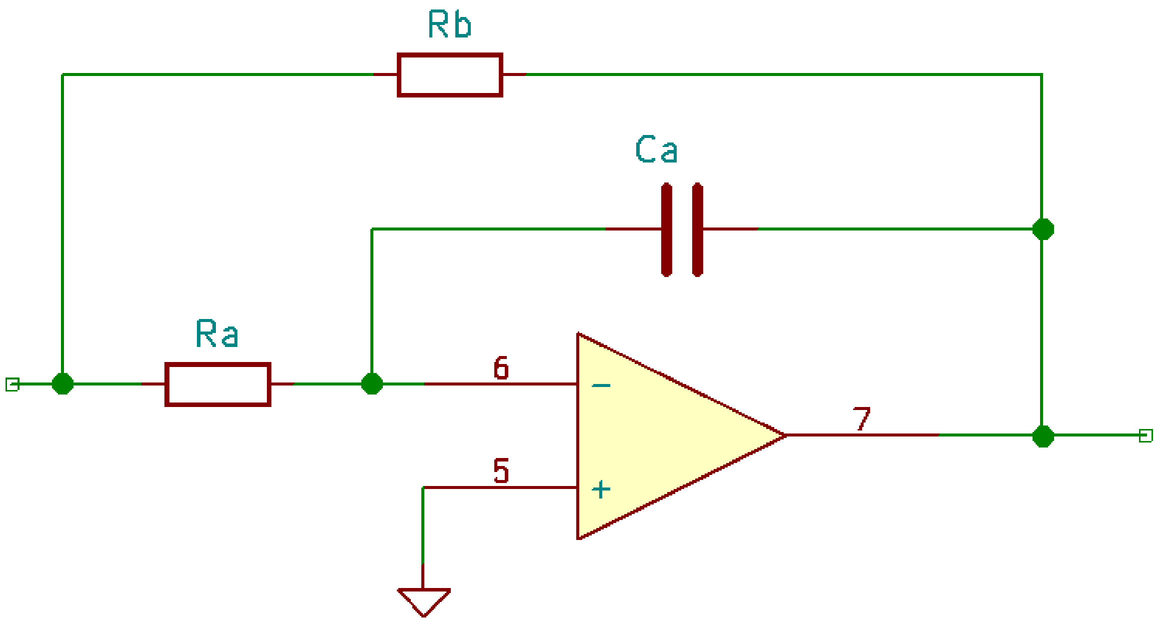

Figure 2: Subcircuit often found in MFB filters

When the desired parallel resistance R_{par} and inductance L are given and C_a is chosen, the required resistor values are

$$R_a = \frac{\frac{L}{C_a R_{par}} \pm \sqrt{\frac{L^2}{C_a^2 R_{par}^2} - 4\frac{L}{C_a}}}{2}$$

$$R_b = \frac{L}{R_a C_a}$$

There are two solutions because swapping the resistors results in the very same impedance.

4. Second-order MFB sections

Adding a resistor R_c and a capacitor C_b to the circuit of section 3 results in a second-order MFB low-pass stage, which is equivalent to an LRC parallel network with $L = R_a R_b C_a$, with $R = R_a$ // R_b // R_c where // stands for in parallel with, and with $C = C_b$. Hence,

$$\omega_0^2 = \frac{1}{R_a R_b C_a C_b}$$

$$Q=\frac{\omega_0 C_b}{1/R_a+1/R_b+1/R_c}$$

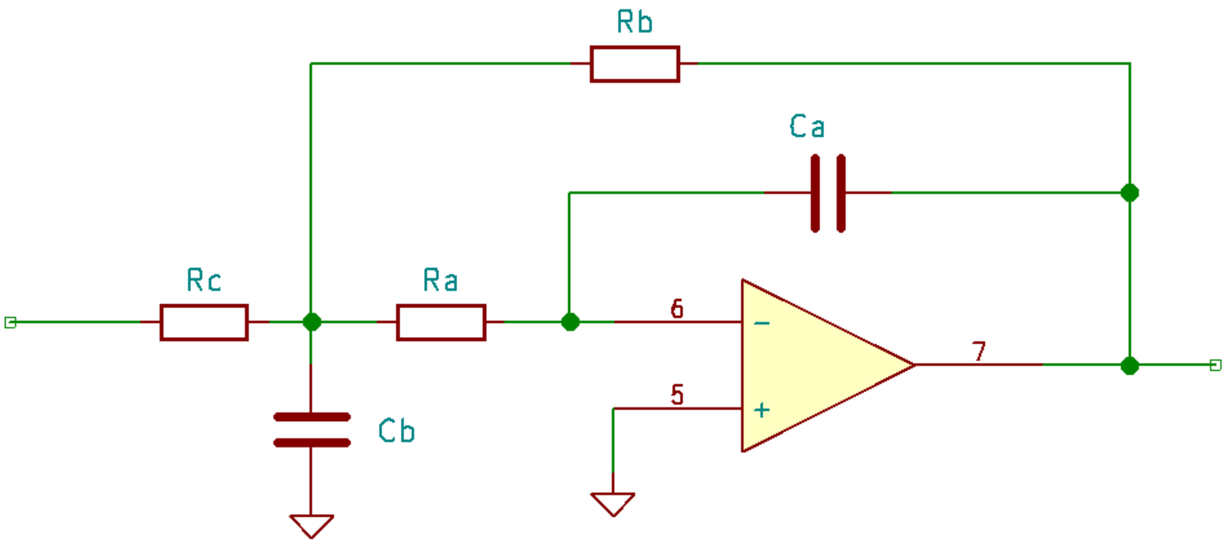

Figure 3: Second-order MFB stage

As capacitors come in fewer standard values than resistors and as the DC gain is $-R_b/R_c$, it is handy to choose the capacitances, the pole positions and the ratio R_b/R_c and to calculate the rest. In the remainder of this section, we will define $A = R_b/R_c$.

Assuming the target poles are a complex conjugate pair, one can calculate $\omega_0^2 = (\mathrm{Re}(p))^2 + (\mathrm{Im}(p))^2$ and $Q = -\omega_0/(2\ \mathrm{Re}(p))$. Given this, $A = R_b/R_c$ and chosen values for C_a and C_b, one can derive that

$$R_b=\frac{\frac{1}{Q}\pm\sqrt{\frac{1}{Q^2}-4\frac{C_a}{C_b}(1+A)}}{2\omega_0 C_a}$$

$$R_c=\frac{R_b}{A}$$

$$R_a=\frac{1}{\omega_0^2 C_a C_b R_b}$$

When complex or negative values are found, the choice of C_a and C_b was not suitable. I haven't checked this for this type of filter, but usually the Q factor becomes most accurate when capacitance ratios are used that only just make the expression under the square root positive, that is the largest ratio that still meets

$$\frac{C_a}{C_b}\leq\frac{1}{4Q^2(1+A)}$$

5. Reduced DC blocking capacitor output stage

The output stage features a DC blocking capacitor that's inside a feedback loop to get away with a relatively small value, at the expense of subsonic peaking at the op-amp output. It has a second-order high-pass response. The schematic is shown in figure 4. R_2 represents the parallel connection of a resistor that keeps the output biased at 0 V and the load.

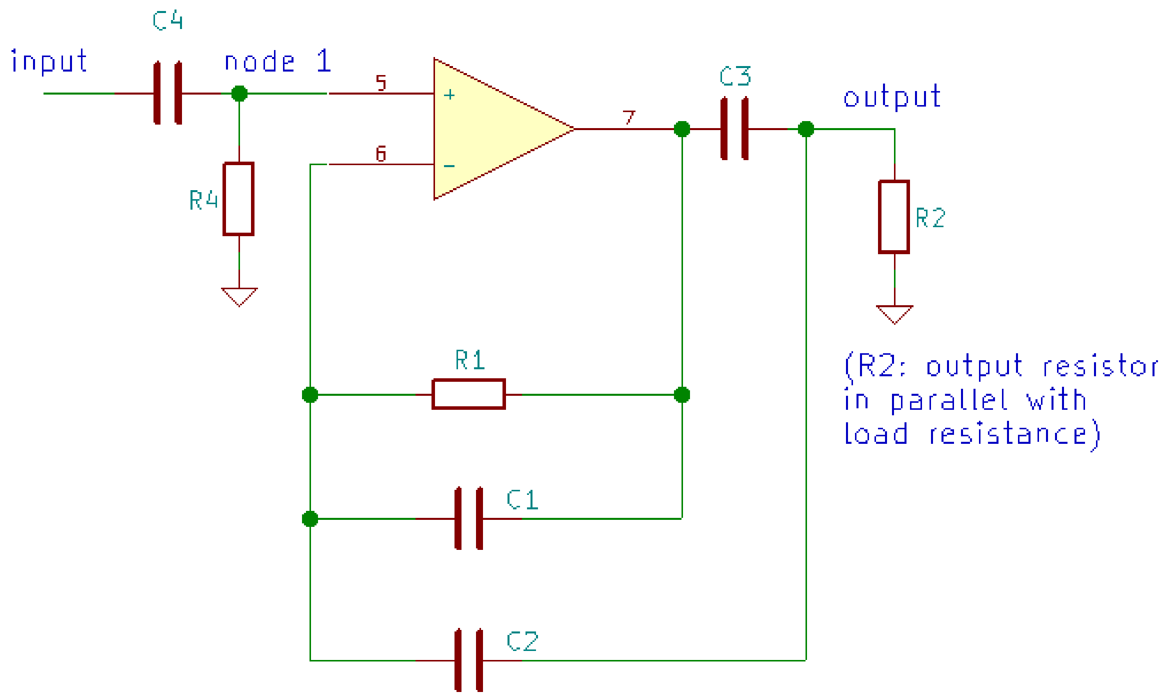

Figure 4: Output stage

Skipping R_4 and C_4 for the time being and using modified nodal analysis to calculate the transfer from node 1 to the output results in

$$\frac{V_{out}}{V_1}=\frac{sR_2(C_2+C_3)\left(sR_1\frac{C_1C_3+C_2C_3+C_1C_2}{C_2+C_3}+1\right)}{s^2(C_1C_3+C_2C_3+C_1C_2)R_1R_2+s(R_1C_1+R_2C_3+R_2C_2)+1}$$

The numerator shows that there is one zero in the origin and one negative real zero, while a normal second-order high-pass has two zeros in the origin. This can be corrected for with R_4 and C_4 by choosing

$$R_4C_4=R_1\frac{C_1C_3+C_2C_3+C_1C_2}{C_2+C_3}$$

As R_2 is not in the equation, this correction will work for any load resistance.

The denominator of the transfer shows that there are two poles with

$$\omega_0=\frac{1}{\sqrt{R_1R_2(C_1C_3+C_2C_3+C_1C_2)}}$$

$$Q=\frac{1}{\omega_0(R_1C_1+R_2C_3+R_2C_2)}=\frac{\sqrt{R_1R_2(C_1C_3+C_2C_3+C_1C_2)}}{R_1C_1+R_2(C_2+C_3)}$$

When $R_2 = \frac{R_1 C_1}{C_2 + C_3}$, both the numerator and the denominator of Q have a sensitivity of ½ to R_2. That is, there is an optimum in Q as a function of R_2 for this value of R_2. This optimum is actually a maximum.

Filling in $R_2 = \frac{R_1 C_1}{C_2 + C_3}$ in the expression for Q and rearranging terms,

$$Q_{max} = \frac{1}{2}\sqrt{\frac{C_1 C_3 + C_2 C_3 + C_1 C_2}{C_1 C_2 + C_1 C_3}} = \frac{1}{2}\sqrt{\frac{C_1 + \frac{C_2 C_3}{C_2 + C_3}}{C_1}}$$

If you want to prevent subsonic peaking across the load for any load resistance, Q_{max} has to be smaller than or equal to ½√2, as an optimally flat second-order high-pass has a Q of precisely ½√2. This is met when $\frac{C_2 C_3}{C_2 + C_3} \le C_1$, so the capacitance of the series connection of C_2 and C_3 has to be smaller than or equal to C_1 to prevent peaking across the load for any load resistance (preferably equal, if you want the response to be as flat as possible under this constraint).

If you don't mind a small amount of subsonic peaking at some load resistances, Q_{max} can be made a bit larger, for example 1. This is met when $\frac{C_2 C_3}{C_2 + C_3} = 3 C_1$, so the capacitance of the series connection of C_2 and C_3 can then be up to three times C_1. I've used a ratio of two in my DAC, so it is somewhere in between with a maximum Q of ½√3.

6. Combining an LC filter with an MFB stage

Doing measurements with 1 kHz, -60 dB DSD test signals, bohrok2610 found some low-level distortion components that later turned out to be intermodulation products between spectral peaks around half the sample frequency or its odd multiples. He noticed that the levels of the intermodulation products depended on the output filter, so apparently these products are (largely or completely?) generated in the output filter. Even though their level was already quite low, around -130 dB DSD (some 20 dB less than similar intermodulation products of the DSC 2.5.2), it would be nice to suppress them even further.

As half the sample frequency means 1.4112 MHz for DSD64, 11.2896 MHz for DSD512, a typical audio op-amp or discrete feedback amplifier made of typical audio transistors will not have much loop gain left at these frequencies. Adding more passive filtering before the first active part of the filter was therefore the logical approach. Realizing all filtering this way has the advantage that one active stage can be skipped.

The proposed configuration (first suggested by ThorstenL) is shown in Figure 5.

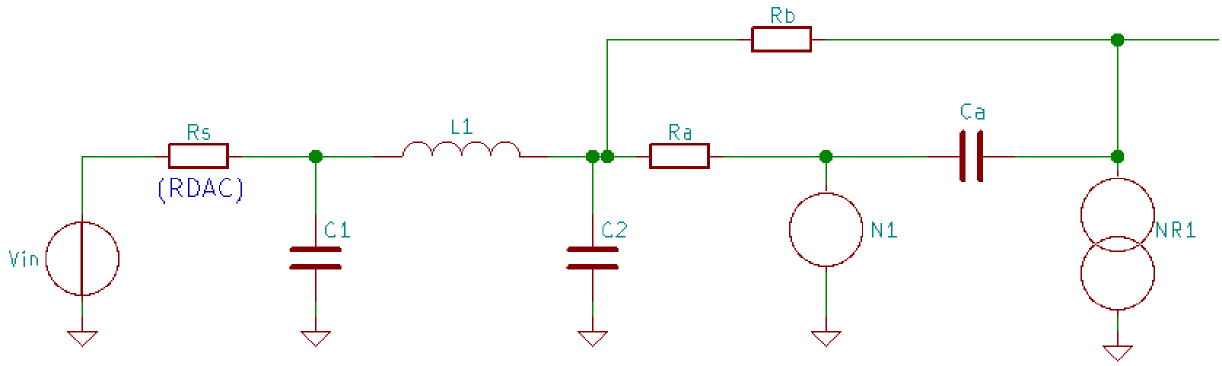

Figure 5: Combination of an LC filter and an MFB stage. The single circle is a nullator, a network element that has zero voltage across it and conducts zero current, basically a model of an ideal op-amp input stage. The double circle is a norator, a network element that can conduct any current and have any voltage across it, basically a model for an ideal op-amp output stage.

Figure 6 shows the exact same using an op-amp symbol rather than a separate nullator and norator.

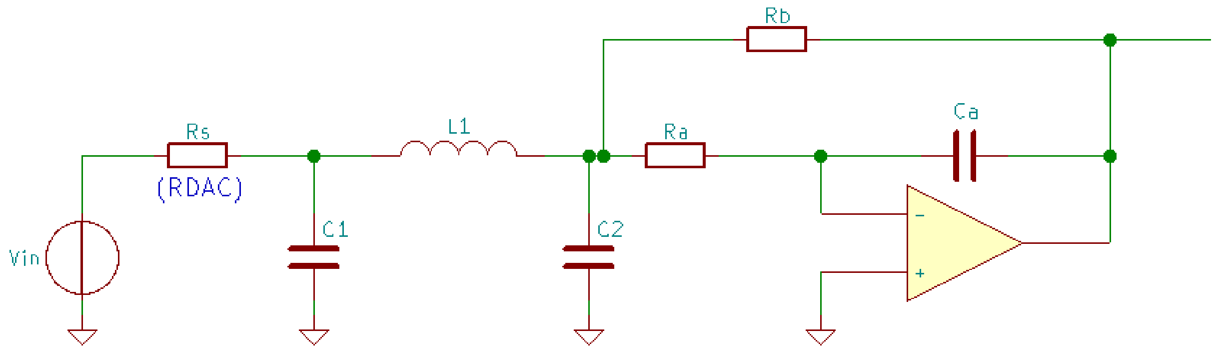

Figure 6: Same circuit drawn with an (ideal) op-amp

A disadvantage of separate nullators and norators compared to ideal op-amps is that only few people are used to the symbols.

Advantages of separate nullators and norators over ideal op-amps are that you see where the current through the output goes to - the current flowing into the output has to come out somewhere, but a standard op-amp symbol doesn't show that - and for circuits with more than one nullator and norator, that you see that there are different ways to group the nullators and norators. In fact there is no need to group them at all at the nullator/norator abstraction level, the number of nullators just needs to be the same as the number of norators. Once you want to implement them as feedback amplifiers, you have to decide which nullator and which norator become the input and output stage of what amplifier. Theoretically, you can make any choice you like. Often the most obvious choice is the only practical one, but there are exceptions to that.

Anyway, an advantage of the configuration of Figure 5 / Figure 6 is that the input keeps behaving as a virtual ground at low and high frequencies. There is an impedance bump in between, but the same holds for the input of the original filter. In contrast to that, a simple resistively-terminated LC ladder low-pass filter has an impedance that doesn't go below the value of the termination resistor at low frequencies. Another advantage is that you can make a fourth-order filter with only one inductor (or one inductor per side when you make it balanced).

As explained in section 3, R_a, R_b, C_a and the nullator and norator/ideal op-amp together behave as a parallel connection of an inductance $L = R_a R_b C_a$ and a resistance $R_a R_b/(R_a + R_b)$. The output voltage is proportional to the current flowing into the inductance. The filter is therefore equivalent to Figure 7 with $L_2 = R_a R_b C_a$ and $R_L = R_a R_b/(R_a + R_b)$.

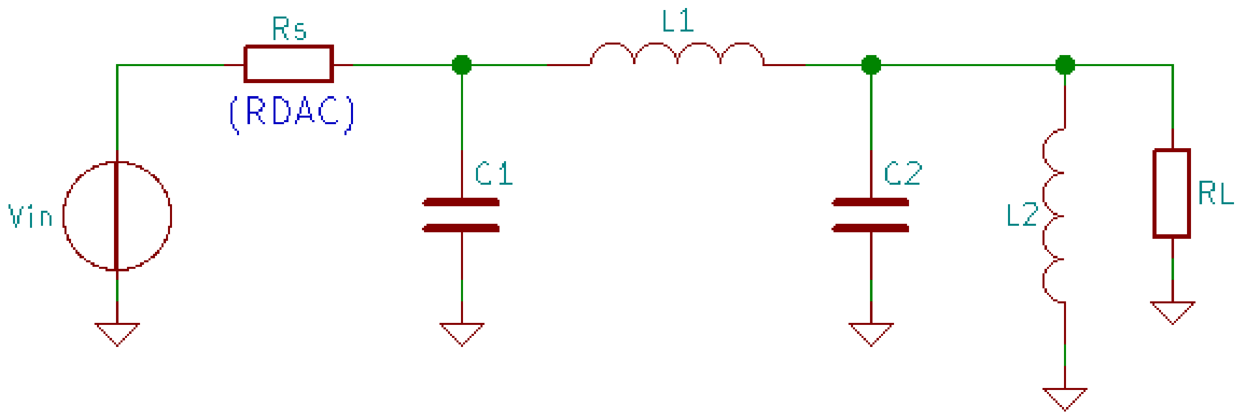

Figure 7: Equivalent ladder filter, the current through L_2 is (proportional to) the output signal.

Figure 7 is not an ordinary LC low-pass filter with the voltage across R_L as output quantity. Therefore, one cannot use a table with normalized LC low-pass filter values and denormalize them. In fact, when the voltage across R_L is regarded as the output quantity, it is an asymmetrical bandpass filter with one zero in the origin, having a normalized transfer function

$$H(s)=K\frac{s}{s^4+a_3 s^3+a_2 s^2+a_1 s+1}$$

Fortunately, procedures to synthesize LC filters with any desired transfer function have been known for decades. As I didn't know them, I borrowed the book of DeVerl S. Humpherys, *The analysis, design, and synthesis of electrical filters*, Prentice-Hall, 1970. Among many other interesting things, Humpherys describes a procedure invented by Darlington in 1939 (the same Darlington who would later patent all ways he could think of to connect two transistors).

You really have to read the book for details, but roughly, it boils down to this. The LC network between the termination resistors (the lossless coupling network) cannot dissipate any power, so all signal power that the source delivers to the LC network has to end up in the load. In transmission line terminology, the only way the LC network can suppress signals is by reflecting them.

From the desired transfer poles and zeros and the desired impedance mismatch at some given frequency, one can calculate the square of the magnitude of the required input reflection coefficient ρ as a function of frequency. Humpherys actually uses a generalized version of that: the product of the reflection coefficient as a function of the Laplace variable s and the same function of -s, that is, $\rho(s)\rho(-s)$.

One can now calculate the numerator polynomial of $\rho(s)\rho(-s)$, which is eight order for a fourth-order filter, and calculate its zeros. I used a polynomial root finding routine of a computer program for that, I don't know how Darlington did it back in 1939.

The eight zeros come in four complex-conjugate pairs, half in the left and half in the right half plane. Half of these zeros have to be assigned to $\rho(s)$. This leads to four different possibilities. The poles of $\rho(s)$ are simply the poles of the desired transfer function.

Once a choice has been made for the zeros of $\rho(s)$, one can calculate the input impedance of the filter. This input impedance can then be used to find the values of the filter. By repeatedly looking at what happens to the immitance (impedance or admittance) for $s \to \infty$ or $s \to 0$, one can find values of series inductances or parallel capacitances, subtract them, take the reciprocal of the rest and continu to the next step. (Apparently it is also possible to simplify the calculations by using an impedance parameter of the unloaded LC coupling network rather than the input impedance with the load resistor at the end, but I didn't manage to get anything useful out of the calculation that way.)

As I wrote, there were four possible choices for the zeros of $\rho(s)$. They all result in the same transfer, but in very different input impedance curves. I chose the one with the nicest looking input impedance curve, that is, the one with the smallest peak.

Using this procedure, I found the normalized values of table 1. (The value for an infinite source impedance was actually found by a simpler method that only produces one solution, also explained in the book.) They are normalized to $R_L = 1$ for easy interpolation.

Table 1. Filter values normalized to $R_L = 1$ *and* $\omega_c = 1$, *fourth-order Butterworth pole positions*

Target for $\lvert Z_{in}\rvert_{max}/R_s$	1/2	1/3	1/4	1/10	0
C_1	0.43340884	0.46644362	0.48412495	0.51774659	0.5411961
L_1	2.501003	2.2809323	2.1717475	1.9767144	1.8477591
C_2	0.41070789	0.40262255	0.39816815	0.38930812	0.38268343
L_2	2.2462372	2.3344838	2.3887317	2.5098366	2.6131259
R_L	1	1	1	1	1
R_s	12.939046	16.563774	20.323142	43.436521	∞

Regarding the "Target for $\lvert Z_{in}\rvert_{max}/R_s$", I determined numerically that the transfer function has a maximum at a normalized radian frequency of 0.87168554 and guessed that with the correct choice of the zeros of ρ, that would also be the frequency of a maximum of the input impedance. I scaled the reflection coefficient such that its value at that frequency corresponded to a 2:1, 3:1 and so on impedance mismatch between the source and the filter input impedance. For some reason, however, the real input impedance maximum is slightly larger than the target and occurs at a slightly different frequency.

Regarding the source impedance, the SN74LV574A has a maximum output resistance of 43.75 Ω when high, 34.375 Ω when low, calculated from the output voltage specifications at 4.5 V supply voltage, ±16 mA output current. As there are on average two outputs high and six low, the maximum weighted average output resistance is 36.71875 Ω. The typical value must be less than that, wild guess about 20 Ω. With the 3.01 kΩ series resistors, the total resistance is then (3010 Ω + 20 Ω)/8 = 378.75 Ω. (Single-ended, I look at only one half of the differential filter for now.)

The original filter has a theoretical input impedance maximum of 198.18 Ω. That's 0.52325 times 378.75 Ω, so the input impedance peak of the original filter is comparable to that of filters according to the column "1/2" of the table.

Attempting to make the input impedance slightly lower without overdoing it (as I didn't get complaints about the present input impedance and as lowering the input impedance aggravates the effect of the input noise voltage of the nullator implementation), I chose the values of the column "1/3". Scaling these to 80 kHz cut-off frequency and R_s = 378.75 Ω leads to these values:

$L_1 \approx$ 103.76141 μH

$C_1 \approx$ 40.582235 nF

$C_2 \approx$ 35.029577 nF

$L_2 \approx$ 106.19752 μH

$R_L \approx$ 22.866166 Ω

The first three values are all just above a convenient standard value. Scaling up the cut-off frequency by the cubic root of the product of the ratios of these values to the nearest standard value leads to:

$f_c \approx$ 83.720674 kHz

$L_1 \approx$ 99.1501 μH

$C_1 \approx$ 38.778698 nF

$C_2 \approx$ 33.472809 nF

$L_2 \approx$ 101.47794 μH

$R_L \approx$ 22.866166 Ω

The first three component values are now very close to 100 μH, 39 nF and 33 nF.

L_2 and R_L are to be realized with an MFB structure. With an ideal op-amp and keeping the gain the same as for the original filter, a feedback resistor of 845 Ω, an input resistor of 23.502147 Ω and an integration capacitor of 5.109841 nF will do the trick. The capacitance is quite close to the E24 standard value 5.1 nF, but it can also be realized with two E12 capacitors in parallel, namely 3.3 nF and 1.8 nF. The E96 value closest to 23.502147 Ω is 23.7 Ω, but one can also round it down to a slightly smaller value and put a small resistor in series with the integration capacitor for phase compensation, as shownin Figure 1.

Using an ideal op-amp, the result is as shown in Figure 8.

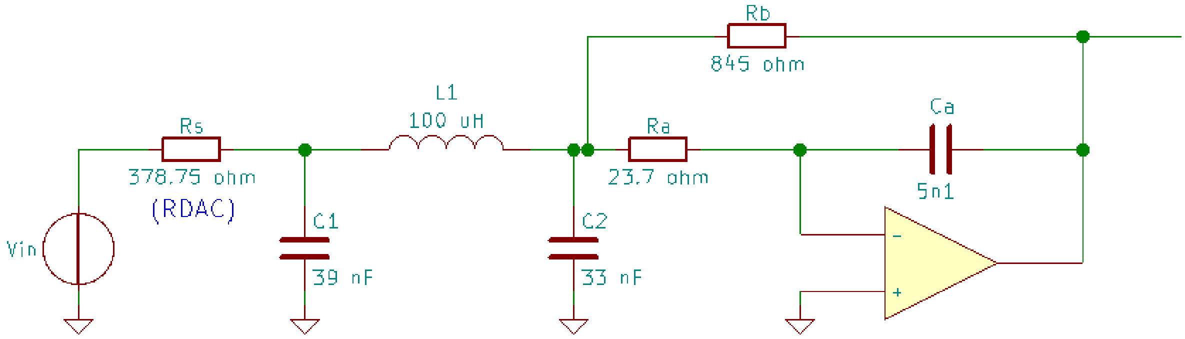

Figure 8: Fourth-order Butterworth filter at 83.72 kHz. *In case of a non-ideal op-amp, one can put a phase compensation resistor in series with the feedback capacitor and reduce* R_a *accordingly.*

The input impedance (excluding R_s) turns out to have a broad peak of about 129 Ω at 75 kHz, just above the 378.75 Ω/3 = 126.25 Ω target.