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A new method in highway route design: joining circular arcs by a single C-Bézier curve with shape parameter*

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Abstract: We constructed a single C-Bézier curve with a shape parameter for G^2 joining two circular arcs. It was shown that an S-shaped transition curve, which is able to manage a broader scope about two circle radii than the Bézier curves, has no curvature extrema, while a C-shaped transition curve has a single curvature extremum. Regarding the two kinds of curves, specific algorithms were presented in detail, strict mathematical proofs were given, and the effectiveness of the method was shown by examples. This method has the following three advantages: (1) the pattern is unified; (2) the parameter able to adjust the shape of the transition curve is available; (3) the transition curve is only a single segment, and the algorithm can be formulated as a low order equation to be solved for its positive root. These advantages make the method simple and easy to implement.

Key words: Transition curve, C-Bézier curve, Monotone curvature, Shape parameter
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INTRODUCTION

In many applications such as highway and railway route design (Hartman, 1957; Baass, 1984) and car-like robot path planning (Fleury *et al.*, 1995), it is complicated and important to construct a transition curve of G^2 contact between two circular arcs. An ideal transition curve should not contain any inflection point, and its bending degree should increase/decrease gradually with the arc length. It is a tradition for many years to use clothoid as the transition curve in highway design (Hartman, 1957). However, to use clothoid is not convenient in a computer-aided design (CAD) system, since it is a transcendental function defined in terms of the Fresnel integral. Recently, many researchers have proposed the cubic Bézier curve and the quintic Pythagorean-Hodograph (PH) curve as alternatives in transition curve design (Walton and Meek, 1996a; 1996b; 1999; 2002; Habib, 2004; Habib and Sakai, 2007). In general, a transition curve is composed of two curve segments, for example, two clothoid segments, two cubic Bézier spiral segments, or two PH quintic spiral segments (Meek and Walton, 1989; Walton and Meek, 1996a; 1996b).

Joining two circular arcs with two cubic Bézier or two quintic PH spiral segments instead of two clothoid segments can avoid the processing of the transcendental function in a CAD system. However, in the entire highway alignment design a rational polynomial will be inevitably used. This will introduce the weights and increase the computation load. On the other hand, as the weights in the transition curves are all constant, they cannot be used to adjust the curve shape. Moreover, this will cause some designing trouble and raise the engineering cost.

It is important to know whether there is another kind of transition curve with a shape parameter suitable for joining the circular arcs, and whether this kind of transition curve can be composed of only a

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single curve segment. As is well known, the Bézier curve cannot express the conic section precisely in current CAD systems. To avoid this constraint, the C-Bézier curve was proposed (Zhang, 1996). The C-Bézier curve, provided with a shape parameter, can express precisely the conic sections including the circular arcs and free form of the curve. So it can be used as the transition curve for road design. It is worthwhile to unify the circular arcs and the transition curves under the C-Bézier pattern, and to adjust the shape of the transition curve using their shape parameters while preserving the required geometric features. Since it is obviously more convenient using a single curve segment instead of two segments to represent the transition curve, it is also worthwhile to explore the feasibility of adopting a single C-Bézier transition curve segment. Actually, a single cubic Bézier or single quintic PH spiral segment instead of two segments has been used as the transition curve (Walton and Meek, 1999; 2002; Habib, 2004; Habib and Sakai, 2007).

This paper is a summing-up to accomplish this task. The algorithms that join two circular arcs by a single fair C-Bézier transition curve with a shape parameter were proposed. There is no curvature extreme point in the S-shaped transition curve, but a curvature extreme point in the C-shaped transition curve. Since the transition curve is a single curve segment with the shape parameter, the design is simple and the curve shape can be adjusted using the shape parameter. More importantly, the entire highway alignment design can be unified under the C-Bézier model, and has a broader applicable scope than that using the Bézier curve.

PRELIMINARIES

Notations and conventions

All points and vectors in the plane are represented by boldface, e.g.,

$$\boldsymbol{P}=\begin{pmatrix}p_x\\p_y\end{pmatrix},\quad \boldsymbol{Q}=\begin{pmatrix}q_x\\q_y\end{pmatrix},$$

and the length of a vector $\boldsymbol{P}$ is denoted by

$$\|\boldsymbol{P}\|=\sqrt{p_x^{\,2}+p_y^{\,2}}.$$

Let θ be the counterclockwise directed angle from vector $\boldsymbol{P}$ to vector $\boldsymbol{Q}$. Suppose all angles measured counterclockwise take positive values, otherwise negative. Define

$$\begin{aligned}\boldsymbol{P}\times\boldsymbol{Q}&=p_xq_y-p_yq_x=\|\boldsymbol{P}\|\|\boldsymbol{Q}\|\sin\theta,\\ \boldsymbol{P}\cdot\boldsymbol{Q}&=p_xq_x+p_yq_y=\|\boldsymbol{P}\|\|\boldsymbol{Q}\|\cos\theta.\end{aligned}$$

Then the signed curvature $\kappa(t)$ and its derivative $\kappa'(t)$ of a parametric curve $\boldsymbol{P}(t)$ in the plane are

$$\kappa(t)=\frac{\boldsymbol{P}'(t)\times\boldsymbol{P}''(t)}{\|\boldsymbol{P}'(t)\|^3},\ \ \kappa'(t)=\frac{\phi(t)}{\|\boldsymbol{P}'(t)\|^5},\tag{1}$$

where

$$\begin{aligned}\phi(t)=&\frac{\mathrm{d}(\boldsymbol{P}'(t)\times\boldsymbol{P}''(t))}{\mathrm{d}t}(\boldsymbol{P}'(t)\cdot\boldsymbol{P}'(t))\\ &-\frac{3}{2}(\boldsymbol{P}'(t)\times\boldsymbol{P}''(t))\frac{\mathrm{d}(\boldsymbol{P}'(t)\cdot\boldsymbol{P}'(t))}{\mathrm{d}t}.\end{aligned}\tag{2}$$

Therefore, when one traverses the curve along the direction of an increasing parameter, the curvature is positive if the center of the curvature is on his/her left side; otherwise, the curvature is negative. The spiral refers to a curve with monotone curvature of constant sign (Guggenheimer, 1963).

Cubic C-Bézier curve

A cubic C-Bézier curve that takes $\{Z_i(t)\}_{i=0}^3$ as the basis and $\{\boldsymbol{P}_i\}_{i=0}^3$ as control points in the interval [0, π/2] is defined as

$$\begin{aligned}\boldsymbol{P}(t)&=Z_0(t)\boldsymbol{P}_0+Z_1(t)\boldsymbol{P}_1+Z_2(t)\boldsymbol{P}_2+Z_3(t)\boldsymbol{P}_3\\ &=\frac{2}{\pi-2}\begin{bmatrix}\sin t\\ \cos t\\ t\\ 1\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}0 & \frac{2-\pi}{4-\pi} & \frac{2}{4-\pi} & -1\\ -1 & \frac{2}{4-\pi} & \frac{2-\pi}{4-\pi} & 0\\ -1 & \frac{2}{4-\pi} & \frac{-2}{4-\pi} & 1\\ \frac{\pi}{2} & \frac{-2}{4-\pi} & \frac{\pi-2}{4-\pi} & 0\end{bmatrix}\begin{pmatrix}\boldsymbol{P}_0\\ \boldsymbol{P}_1\\ \boldsymbol{P}_2\\ \boldsymbol{P}_3\end{pmatrix}.\end{aligned}\tag{3}$$

It is easy to obtain the first and second derivatives of the C-Bézier curve $\boldsymbol{P}(t)$ as follows:

$$\boldsymbol{P}'(t)=\frac{2}{\pi-2}[(1-\sin t)(\boldsymbol{P}_1-\boldsymbol{P}_0)+(1-\cos t)(\boldsymbol{P}_3-\boldsymbol{P}_2)]+\frac{2}{4-\pi}(\cos t+\sin t-1)(\boldsymbol{P}_2-\boldsymbol{P}_1),$$

$$\boldsymbol{P}''(t)=\frac{2}{\pi-2}\left[\left(\frac{\pi-2}{4-\pi}(\boldsymbol{P}_2-\boldsymbol{P}_1)-(\boldsymbol{P}_1-\boldsymbol{P}_0)\right)\cos t+\left(\boldsymbol{P}_3-\boldsymbol{P}_2-\frac{\pi-2}{4-\pi}(\boldsymbol{P}_2-\boldsymbol{P}_1)\right)\sin t\right].$$

Let the length of the vector $\boldsymbol{P}_{i+1}\boldsymbol{P}_i$ be l_i, the corresponding unit vector be $\boldsymbol{T}_i$, and the corresponding unit normal vector be $\boldsymbol{N}_i$ in the right-hand coordinate system, which satisfy

$$l_i\boldsymbol{T}_i=\boldsymbol{P}_{i+1}-\boldsymbol{P}_i,\ \ i=0,\ 1,\ 2.$$

Let θ be the directed angle from $\boldsymbol{T}_0$ to $\boldsymbol{T}_1$ and φ be the angle from $\boldsymbol{T}_1$ to $\boldsymbol{T}_2$. It follows that

$$\boldsymbol{T}_0\times\boldsymbol{T}_1=\sin\theta,\ \boldsymbol{T}_1\times\boldsymbol{T}_2=\sin\varphi,\ \boldsymbol{T}_0\times\boldsymbol{T}_2=\sin(\theta+\varphi);$$
$$\boldsymbol{T}_0\cdot\boldsymbol{T}_1=\cos\theta,\ \boldsymbol{T}_1\cdot\boldsymbol{T}_2=\cos\varphi,\ \boldsymbol{T}_0\cdot\boldsymbol{T}_2=\cos(\theta+\varphi).$$

Then, the curvatures at two endpoints of the curve $\boldsymbol{P}(t)$ are

$$\kappa(0)=\frac{(\pi/2-1)^2l_1}{(2-\pi/2)l_0^2}\sin\theta,\ \ \kappa(\pi/2)=\frac{(\pi/2-1)^2l_1}{(2-\pi/2)l_2^2}\sin\varphi. \tag{4}$$

CONSTRUCTING C-BÉZIER TRANSITION CURVE BETWEEN TWO CIRCULAR ARCS

Given two non-enclosing circles $\boldsymbol{\Omega}_i$ (i=0, 1), with the centres $\boldsymbol{C}_i$ and radii r_i. Our research is aimed at finding a single C-Bézier curve as the transition curve. Without loss of generality, assume $r_0\geq r_1$, and define

$$r=\|\boldsymbol{C}_1-\boldsymbol{C}_0\|,\ \boldsymbol{C}_1-\boldsymbol{C}_0=\begin{pmatrix}c_x\\ c_y\end{pmatrix},\ \lambda=\sqrt{\frac{r_1}{r_0}}.$$

As in (Walton and Meek, 1999; Habib, 2004), the angle conditions for the S-shaped transition curve can be taken as (Fig.1)

$$\varphi=-\theta,\ \ 0<\theta<\pi/2. \tag{5}$$

Then the directed curvatures at the two endpoints of curve $\boldsymbol{P}(t)$ are $1/r_0$ and $-1/r_1$, respectively. Also, the angle conditions for the C-shaped transition curve can be taken as (Fig.2)

$$\varphi=\theta,\ \ 0<\theta<\pi/2. \tag{6}$$

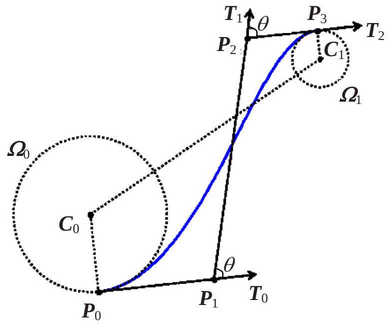

Fig.1 An S-shaped C-Bézier transition curve between two circular arcs

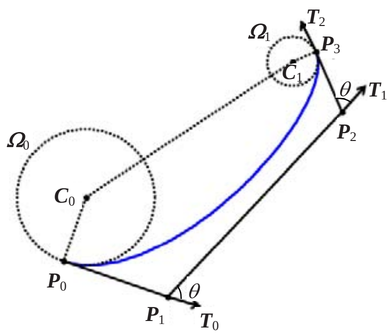

Fig.2 A C-shaped C-Bézier transition curve between two circular arcs

Then the directed curvatures at the two endpoints of curve $\boldsymbol{P}(t)$ are $1/r_0$ and $1/r_1$, respectively. Using the curvature Eq.(4) of curve $\boldsymbol{P}(t)$ in the above two cases, the length of each edge of the control polygon can be obtained as follows:

$$\begin{cases}l_0=(\pi/2-1)mr_0\tan\theta,\\ l_1=(2-\pi/2)m^2r_0\tan\theta\sec\theta,\ \ m>0.\\ l_2=\lambda l_0,\end{cases} \tag{7}$$

Here m is the shape parameter. Then the first and second derivatives of $\boldsymbol{P}(t)$ are

$$\boldsymbol{P}'(t)=mr_0\tan\theta\cdot[(1-\sin t)\boldsymbol{T}_0+m(\cos t+\sin t-1)\sec\theta\cdot\boldsymbol{T}_1+\lambda(1-\cos t)\boldsymbol{T}_2], \tag{8}$$

$$\boldsymbol{P}''(t)=mr_0\tan\theta\cdot[-\cos t\cdot\boldsymbol{T}_0+m(\cos t-\sin t)\cdot\sec\theta\cdot\boldsymbol{T}_1+\lambda\sin t\cdot\boldsymbol{T}_2]. \tag{9}$$

S-shaped transition curve

Lemma 1 Suppose that the angle condition Eq.(5) holds. If (1) $m\geq 2/3$, $1/3\leq\lambda\leq 1$ or (2) $m\geq 1$, $1/7\leq\lambda\leq 1$, then the curvature of the cubic C-Bézier curve $\boldsymbol{P}(t)$ monotonically decreases in the interval [0, π/2], and hence the following condition

$$\phi(t)<0,\ \ 0\leq t\leq \pi/2 \tag{10}$$

for the S-shaped transition curve can be met.

The proof of Lemma 1 is given in Appendix A. It should be noted that the condition $r_1\leq r_0\leq 49r_1$ ($1/7\leq\lambda\leq 1$) has a broader scope than the condition $r_1\leq r_0\leq 36r_1$ in (Habib, 2004). Also, since $1/7<3\sqrt{3}/(10\sqrt{10})$, the condition has a broader scope than $3\sqrt{3}/(10\sqrt{10})\leq\lambda\leq 1$ in (Habib and Sakai, 2007). Therefore, the C-Bézier curve has a broader applicable scope than the cubic Bézier curve and the quintic PH curve in transition curve design.

Now we discuss how to construct the transition curve using a C-Bézier curve. First note that

$$\boldsymbol{T}_0=\boldsymbol{T}_2,\ \boldsymbol{T}_0\times\boldsymbol{T}_1=-\boldsymbol{T}_1\times\boldsymbol{T}_2=\sin\theta,\ \boldsymbol{T}_0\cdot\boldsymbol{T}_1=-\boldsymbol{T}_1\cdot\boldsymbol{T}_2=\cos\theta.$$

Hence the vectors $\boldsymbol{T}_1$ and $\boldsymbol{N}_1$ can be expressed as

$$\begin{cases}\boldsymbol{T}_1=\boldsymbol{T}_0\cos\theta+\boldsymbol{N}_0\sin\theta,\\ \boldsymbol{N}_1=-\boldsymbol{T}_0\sin\theta+\boldsymbol{N}_0\cos\theta.\end{cases} \tag{11}$$

From the point property in the circle and Eq.(7), we can obtain

$$\boldsymbol{P}(0)-\boldsymbol{C}_0=-r_0\boldsymbol{N}_0,\ \boldsymbol{P}(\pi/2)-\boldsymbol{C}_1=r_1\boldsymbol{N}_2=r_1\boldsymbol{N}_0, \tag{12}$$

$$\begin{aligned}\boldsymbol{P}(\pi/2)-\boldsymbol{P}(0)&=l_0\boldsymbol{T}_0+l_1\boldsymbol{T}_1+l_2\boldsymbol{T}_2\\ &=\alpha_1 m r_0\tan\theta\cdot\boldsymbol{T}_0+\alpha_2\rho\boldsymbol{N}_0,\end{aligned} \tag{13}$$

where

$$\alpha_1=(\pi/2-1)(1+\lambda)+(2-\pi/2)m,$$

$$\alpha_2=(2-\pi/2)/r_0,\ \ \rho=(mr_0\tan\theta)^2.$$

It follows from Eqs.(12) and (13) that

$$\alpha_1 m r_0\tan\theta\cdot\boldsymbol{T}_0+[\alpha_2\rho-(r_1+r_0)]\boldsymbol{N}_0=\boldsymbol{C}_1-\boldsymbol{C}_0. \tag{14}$$

From Eq.(14) we can obtain the identity

$$f_1(\rho)=:\alpha_1^2\rho+[\alpha_2\rho-(r_1+r_0)]^2-r^2=0. \tag{15}$$

If $r_0+r_1<r$, then $f_1(0)=(r_0+r_1)^2-r^2<0$. Since its constant term is negative, the above quadratic Eq.(15) has a unique positive root. By Eq.(14) we can obtain

$$\boldsymbol{T}_0=\{\alpha_1^2\rho+[\alpha_2\rho-(r_1+r_0)]^2\}^{-1}\cdot\begin{pmatrix}\alpha_1 c_x\sqrt{\rho}+[\alpha_2\rho-(r_1+r_0)]c_y\\ \alpha_1 c_y\sqrt{\rho}-[\alpha_2\rho-(r_1+r_0)]c_x\end{pmatrix}.$$

Finally we can obtain the control points of the curve, i.e.,

$$\boldsymbol{P}_0=\boldsymbol{C}_0-r_0\boldsymbol{N}_0,\ \ \boldsymbol{P}_1=\boldsymbol{P}_0+(\pi/2-1)\sqrt{\rho}\boldsymbol{T}_0, \tag{16}$$

$$\boldsymbol{P}_3=\boldsymbol{C}_1+r_1\boldsymbol{N}_0,\ \ \boldsymbol{P}_2=\boldsymbol{P}_3-\lambda(\pi/2-1)\sqrt{\rho}\boldsymbol{T}_0. \tag{17}$$

Theorem 1 If the radii r_0 and r_1 of the circular arcs satisfy the conditions $r_0+r_1<r$ and $1/7\leq\lambda\leq 1$, then for each value of m (≥ 1), there exists a C-Bézier curve as the S-shaped transition curve between the two circular arcs (Fig.1).

C-shaped transition curve

By Kneser's theorem (Guggenheimer, 1963), a C-shaped transition curve cannot be a spiral if one circle is not contained in the other circle; in other words, the curvature of this transition curve cannot be monotone. Therefore, the C-shaped curve must have the interior curvature minimum point. According to the actual request, the C-shaped curve includes only an interior curvature extreme point; that is, $\phi(t)$ in Eq.(2) has and only has a zero point in the interval (0, π/2).

Lemma 2 Suppose the angle condition Eq.(6) holds. If $m\geq(1+\sqrt{7})/3$, then the cubic C-Bézier curve $\boldsymbol{P}(t)$ has and only has one minimum curvature point in the interval (0, π/2).

The proof of Lemma 2 is given in Appendix B. Now we turn to the construction of a C-Bézier curve. First note that

$$\begin{cases}\boldsymbol{T}_0=\boldsymbol{T}_1\cos\theta-\boldsymbol{N}_1\sin\theta,\\ \boldsymbol{N}_0=\boldsymbol{T}_1\sin\theta+\boldsymbol{N}_1\cos\theta,\end{cases} \tag{18}$$

$$\begin{cases}\boldsymbol{T}_2=\boldsymbol{T}_1\cos\theta+\boldsymbol{N}_1\sin\theta,\\ \boldsymbol{N}_2=-\boldsymbol{T}_1\sin\theta+\boldsymbol{N}_1\cos\theta.\end{cases} \tag{19}$$

From the point property in the circle as well as Eqs.(18) and (19), we can obtain

$$\begin{cases} \boldsymbol{P}(0) - \boldsymbol{C}_0 = -r_0 \boldsymbol{N}_0 = -r_0(\boldsymbol{T}_1 \sin\theta + \boldsymbol{N}_1 \cos\theta), \\ \boldsymbol{P}(\pi/2) - \boldsymbol{C}_1 = -r_1 \boldsymbol{N}_2 = -r_1(-\boldsymbol{T}_1 \sin\theta + \boldsymbol{N}_1 \cos\theta). \end{cases} \tag{20}$$

By using Eqs.(7), (18) and (19), we have

$$\begin{aligned} &\boldsymbol{P}(\pi/2) - \boldsymbol{P}(0) = l_0\boldsymbol{T}_0 + l_1\boldsymbol{T}_1 + l_2\boldsymbol{T}_2 \\ &= \tan\theta \cdot (\beta_1 \cos\theta + \beta_2 \sec\theta)\boldsymbol{T}_1 - \beta_3 \tan^2\theta \cos\theta \cdot \boldsymbol{N}_1, \end{aligned} \tag{21}$$

where

$$\beta_1 = (\pi/2 - 1)(1+\lambda)mr_0,\quad \beta_2 = (2-\pi/2)m^2 r_0,$$
$$\beta_3 = (\pi/2 - 1)(1-\lambda)mr_0.$$

It follows from Eqs.(20) and (21) that

$$\begin{aligned} &\tan\theta \cdot \{[\beta_1 - (r_1 + r_0)]\cos\theta + \beta_2 \sec\theta\}\boldsymbol{T}_1 \\ &- \cos\theta \cdot [\beta_3 \tan^2\theta + (r_0 - r_1)]\boldsymbol{N}_1 = \boldsymbol{C}_1 - \boldsymbol{C}_0. \end{aligned} \tag{22}$$

This leads to

$$f_2(\mu) =: \sum_{i=0}^{3} d_i \mu^3 = 0, \tag{23}$$

where

$$\mu = \tan^2\theta,\ d_3 = \beta_2^{\,2},\ d_0 = (r_1 - r_0)^2 - r^2,$$
$$d_2 = \beta_3^{\,2} + 2[\beta_1 - (r_1 + r_0)]\beta_2 + 2\beta_2^{\,2},$$
$$\begin{aligned} d_1 = &[\beta_1 - (r_1 + r_0)]^2 + 2[\beta_1 - (r_1 + r_0)]\beta_2 \\ &+ \beta_2^{\,2} + 2\beta_3(r_1 - r_0) - r^2. \end{aligned}$$

Since β_i>0 (i=1, 2, 3) and $\beta_1-(r_0+r_1)>0$, when $m \geq (1+\sqrt{7})/3$ (i.e., the condition in Lemma 2 holds), if $r_0-r_1<r$, then the sign sequence of the coefficients of the cubic polynomial $f_2(\mu)$ will be (+, +, ?, −), in which "?" means either "+" or "−". That is, the variation of sign of the coefficients of the cubic polynomial $f_2(\mu)$ is equal to 1. Therefore, using the Descartes' rule of signs (Polya and Szego, 2004), we know that the cubic Eq.(23) has a unique positive root μ. Then the vector $\boldsymbol{T}_1$ can be obtained from Eq.(22), and the vectors $\boldsymbol{T}_0$, $\boldsymbol{N}_0$, $\boldsymbol{T}_2$ and $\boldsymbol{N}_2$ can also be determined from Eqs.(18) and (19). Finally the control points $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$ of the curve $\boldsymbol{P}(t)$ can be computed.

Theorem 2 If the radii r_0 and r_1 of the circular arcs satisfy $r_0-r_1<r$, then for each value of m $(\geq (1+\sqrt{7})/3)$, there exists a C-Bézier curve as the C-shaped transition curve between the two circular arcs (Fig.2).

EXAMPLES

This section presents two examples, in which all primary data originate from (Walton and Meek, 1999). Fig.3 presents the cross-section of a cam-like shape, which is composed of two circular arcs joined by an upper and a lower family of C-shaped curves. According to the algorithms presented in the previous section, we obtained the C-shaped curves. The thin C-shaped curves correspond to the shape parameter m=1.22, and the bold ones correspond to m=2.

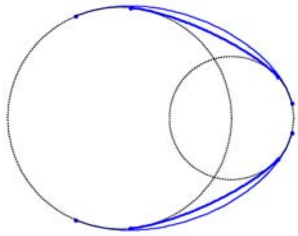

Fig.3 Cam-like cross-section
Thin: m=1.22; bold: m=2

Fig.4 presents the vase profile. The side and base of the vase are represented by a family of S- and C-shaped curves, respectively. The thin curves correspond to the shape parameter m=1.22 (C-shaped) and m=0.7 (S-shaped), and the bold ones correspond to m=5 (S- and C-shaped).

Apparently, the shape parameter m allows an interactive alteration of the curve shape while preserving required geometric features. Therefore, the user may also choose other shape parameters according to the actual need. As can be seen, with the increase of m the arc length of the transition curve between two circles decreases and the transition curve becomes tighter and tends to a line segment within the limit.

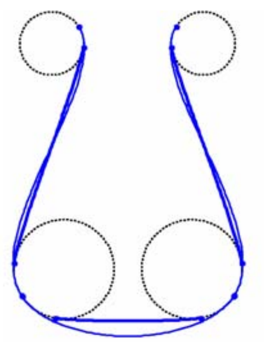

Fig.4 Vase profile
Thin (C-shaped, m=1.22; S-shaped, m=0.7); bold (C- and S-shaped, m=5)

CONCLUSION

The use of a single fair C-Bézier curve for G^2 transition between two circular arcs has been demonstrated. It was shown that an S-shaped transition curve has no curvature extrema, while a C-shaped transition curve has a single curvature extremum. The designer can adjust the shape of the transition curve using the shape parameter *m* while preserving required geometric features. Moreover, the C-Bézier curve has a broader applicable scope than the cubic Bézier curve and the quintic PH curve. On the other hand, the C-Bézier curve can express precisely circular arcs. Therefore, the entire highway alignment design can be unified under the C-Bézier model.

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APPENDIX A: PROOF OF LEMMA 1

From Eqs.(8) and (9), we obtain

$$\boldsymbol{P}'(t)\times\boldsymbol{P}''(t)=m^3r_0^2\tan^3\theta\cdot f_1(t),$$
$$\boldsymbol{P}'(t)\cdot\boldsymbol{P}'(t)=m^2r_0^2\tan^2\theta\cdot f_2(t),$$

where

$$f_1(t)=-\sin t+\lambda\cos t-\lambda+1,$$
$$f_2(t)=[(1-\sin t)+m(\cos t+\sin t-1)+\lambda(1-\cos t)]^2 + m^2\tan^2\theta\cdot(\cos t+\sin t-1)^2.$$

Therefore, Eq.(2) can be expressed as

$$\phi(t)=m^5r_0^4\tan^5\theta\cdot\phi_1(t),$$

where

$$\phi_1(t)=: f_1'(t)f_2(t)-\frac{3}{2}f_1(t)f_2'(t).$$

So, to prove that the curvature of the cubic C-Bézier curve $\boldsymbol{P}(t)$ is monotonically decreasing is equivalent to holding the condition

$$\phi_1(t)<0,\ \ 0<t<\pi/2. \tag{A1}$$

Since $\phi_1(t)$ includes the high degree terms of trigonometric functions, it is very difficult to seek the necessary and sufficient condition for Eq.(A1). So we will seek a sufficient condition instead. For this, a transformation is introduced and expressed as

$$\cos t=\frac{1-u^2}{1+u^2},\ \sin t=\frac{2u}{1+u^2},\ 0<u<1, \tag{A2}$$

and let

$$u=1/(1+s),\ 0<s<+\infty. \tag{A3}$$

By using the software Maple™, under the above transformation, $\phi_1(t)$ can be turned into

$$g(s)=\frac{-1}{(2+2s+s^2)^3}[2sg_1(s)m^2\tan^2\theta + (s^2+2ms+2\lambda)g_0(s)],$$

where

$$g_0(s)=mg_1(s)+g_2(s), \tag{A4}$$

$$\begin{cases} g_1(s)=3s^4+2s^3+(-2\lambda-2)s^2+4\lambda s+12\lambda, \\ g_2(s)=-2s^4+(-4+8\lambda)s^3+16\lambda s^2 \\ \qquad +(16\lambda-8\lambda^2)s-8\lambda^2. \end{cases} \tag{A5}$$

Therefore, a sufficient condition of Eq.(A1) is equivalent to finding an appropriate domain of λ and m, in which the following inequalities hold:

$$g_0(s)>0,\ g_1(s)>0\ (0<s<+\infty).$$

Now we first take a look at $g_1(s)$. When $0<s\leq1$, we have $s^2\leq s\leq1$, and thus

$$g_1(s)>(-2\lambda-2+4\lambda+12\lambda)s^2=(14\lambda-2)s^2.$$

Also when $s>1$, we have $s^4\geq s^3\geq s^2$, and thus

$$g_1(s)>(3+2-2\lambda-2)s^2=(3-2\lambda)s^2.$$

So, if

$$1/7\leq\lambda\leq1, \tag{A6}$$

the inequality $g_1(s)>0$ $(0<s<+\infty)$ holds.

Next turn to $g_0(s)$. Noting that the coefficient of the highest degree term of $g_0(s)$ is $3m-2$, we can see that $m\geq2/3$ is the necessary condition for $g_0(s)$ to be positive $(0<s<+\infty)$. Therefore when Eq.(A6) holds, we can obtain

$$\begin{aligned} g_0(s)&\geq\frac{2}{3}g_1(s)+g_2(s) \\ &=(-8/3+8\lambda)s^3+(44\lambda/3-4/3)s^2 \\ &\quad+(56\lambda/3-8\lambda^2)s+8\lambda-8\lambda^2. \end{aligned}$$

From the above equation, it is easy to know that if $1/3\leq\lambda\leq1$, then $g_0(s)>0$ $(0<s<+\infty)$. Here the domain of λ is smaller than that in Eq.(A6). Therefore suitably enlarging the minimum value of m, for example $m\geq1$, we have

$$\begin{aligned} g_0(s)&\geq g_1(s)+g_2(s) \\ &=s^4+(8\lambda-2)s^3+(14\lambda-2)s^2 \\ &\quad+(20\lambda-8\lambda^2)s+12\lambda-8\lambda^2. \end{aligned}$$

When Eq.(A6) holds, we have $14\lambda-2\geq0$, $20\lambda-8\lambda^2\geq0$, and $12\lambda-8\lambda^2\geq0$; also if $0<s\leq1$, then $s^3\leq s^2\leq s$. Thus

$$\begin{aligned} g_0(s)&>(8\lambda-2+14\lambda-2+20\lambda-8\lambda^2+12\lambda-8\lambda^2)s^3 \\ &=(44\lambda-16\lambda^2-2)s^3\geq(28\lambda-2)s^3\geq2s^3>0. \end{aligned}$$

On the other hand, if $s>1$, then

$$g_0(s)>(1+8\lambda-2)s^3=(8\lambda-1)s^3>0\,.$$

Therefore Lemma 1 is proved.

APPENDIX B: PROOF OF LEMMA 2

From Eqs.(8) and (9), we obtain

$$\begin{aligned} \boldsymbol{P}'(t)\times\boldsymbol{P}''(t)&=m^2r_0^{\,2}\tan^3\theta\cdot f_3(t), \\ \boldsymbol{P}'(t)\cdot\boldsymbol{P}'(t)&=m^2r_0^{\,2}\tan^2\theta\cdot f_4(t), \end{aligned}$$

where

$$\begin{aligned} f_3(t)&=2\lambda\cos^2\theta\cdot(\cos t+\sin t-1) \\ &\quad+m(1-\sin t)+m\lambda(1-\cos t), \\ f_4(t)&=[(1-\sin t)\cos\theta+m(\cos t+\sin t-1)\sec\theta \\ &\quad+\lambda(1-\cos t)\cos\theta]^2 \\ &\quad+[-(1-\sin t)\sin\theta+\lambda(1-\cos t)\sin\theta]^2. \end{aligned}$$

Therefore Eq.(2) can be expressed as

$$\phi(t)=m^4r_0^{\,4}\tan^5\theta\cdot\phi_2(t),$$

where

$$\phi_2(t)=:f_3'(t)f_4(t)-\frac{3}{2}f_3(t)f_4'(t).$$

In order to make $\boldsymbol{P}(t)$ have only a single curvature extremum in the interval $(0, \pi/2)$, the derivatives of its curvature at two endpoints should satisfy

$$\kappa'(0)\leq0,\quad \kappa'(\pi/2)\geq0.$$

This is equivalent to

$$\phi_2(0)=-3m^2+2\lambda\cos^2\theta+2m\leq0, \tag{B1}$$

$$\phi_2(\pi/2)=\lambda^2(3m^2-2\lambda\cos^2\theta-2m\lambda)\geq0. \tag{B2}$$

It can be observed that if Eq.(B1) is satisfied, Eq.(B2) is also satisfied. Solving Eq.(B1) we obtain

$$m \geq (1+\sqrt{7})/3. \tag{B3}$$

Therefore, if Eq.(B3) holds, then Eqs.(B1) and (B2) hold when $\theta \in (0, \pi/2)$, $\lambda \in (0, 1)$.

In order to prove that $\phi_2(t)$ has and only has a zero point in the interval $(0, \pi/2)$ when Eq.(B3) holds, we introduce the transformation formed by Eqs.(A2) and (A3). Using Maple™, $\phi_2(t)$ can be turned into

$$h(s) = \frac{-h_1(s)}{(2+2s+s^2)^3 \cos^2\theta}, \ \ h_1(s) =: \sum_{i=0}^{6} a_i s^i, \ \ 0 < s < +\infty,$$

where

$$\begin{aligned}
a_0 &= -8\lambda^2 \cos^2\theta \cdot (3m^2 - 2\lambda m - 2\lambda \cos^2\theta), \\
a_1 &= -8\lambda[(-6\lambda^2 + 8\lambda m)\cos^4\theta \\
&\quad + (-\lambda m^2 - 2m\lambda^2 - 4\lambda m)\cos^2\theta + 3m^3], \\
a_2 &= -4\lambda[16\lambda\cos^6\theta + (-10\lambda^2 - 8\lambda m - 8\lambda)\cos^4\theta \\
&\quad + (16m^2 - 6\lambda m - 5\lambda m^2)\cos^2\theta + 2m^3], \\
a_3 &= -4m[16\lambda(1-\lambda)\cos^4\theta \\
&\quad + 2\lambda(1-\lambda)\cos^2\theta + (1-\lambda)m^2], \\
a_4 &= 32\lambda^2\cos^6\theta + (-16\lambda^2 - 16\lambda m - 20\lambda)\cos^4\theta \\
&\quad + (-12\lambda m - 10m^2 + 32\lambda m^2)\cos^2\theta + 4m^3, \\
a_5 &= (-12\lambda + 16\lambda m)\cos^4\theta \\
&\quad + (-4m - 2m^2 - 8\lambda m)\cos^2\theta + 6m^3, \\
a_6 &= \cos^2\theta \cdot (3m^2 - 2m - 2\lambda\cos^2\theta).
\end{aligned}$$

Since Eq.(B3) holds, we can obtain $a_0 \leq 0$, $a_6 \geq 0$. At the same time we can see $a_3<0$. Then, we can judge the signs of the coefficients a_1 and a_5. Let

$$\begin{aligned}
b_1 &= (-6\lambda^2 + 8\lambda m)\cos^4\theta + 3m^3 \\
&\quad + (-\lambda m^2 - 2m\lambda^2 - 4\lambda m)\cos^2\theta.
\end{aligned}$$

This is a quadratic trinomial about $\cos^2\theta$. Since $-6\lambda^2+8\lambda m>0$, the minimum value of b_1 is

$$\begin{aligned}
&\frac{\lambda m^2[(96-\lambda)m^2 + (-4\lambda^2 - 80\lambda)m - 4\lambda^3 - 16\lambda^2 - 16\lambda]}{4(-6\lambda^2 + 8\lambda m)} \\
&\geq \frac{\lambda m^2(95m^2 - 84m - 36)}{4(-6\lambda^2 + 8\lambda m)} = \frac{\lambda m^2(5m-6)(19m+6)}{4(-6\lambda^2 + 8\lambda m)}.
\end{aligned}$$

Since $(1+\sqrt{7})/3 > 6/5$, we obtain $b_1>0$, namely $a_1=-8\lambda b_1<0$. At the same time, since

$$-4m - 2m^2 - 8\lambda m \leq 2(-2m\lambda^2 - \lambda m^2 - 4\lambda m) < 0,$$

we obtain $a_5 \geq 2b_1 > 0$. Next we judge the sign of the coefficient a_2. Let

$$\begin{aligned}
c_2 &= 16\lambda\cos^4\theta + (-10\lambda^2 - 8\lambda m - 8\lambda)\cos^2\theta \\
&\quad + (16m^2 - 6\lambda m - 5\lambda m^2 + 2m^3).
\end{aligned}$$

Hence

$$\begin{aligned}
b_2 &= 16\lambda\cos^6\theta + (-10\lambda^2 - 8\lambda m - 8\lambda)\cos^4\theta \\
&\quad + (16m^2 - 6\lambda m - 5\lambda m^2)\cos^2\theta + 2m^3 \\
&\geq c_2\cos^2\theta.
\end{aligned}$$

Since c_2 is a quadratic trinomial about $\cos^2\theta$, noting that $-6\lambda^2+8\lambda m>0$, the minimum value of c_2 is

$$\begin{aligned}
&\frac{1}{64\lambda}[64\lambda(16m^2 - 6\lambda m - 5\lambda m^2 + 2m^3) \\
&\quad - (-10\lambda^2 - 8\lambda m - 8\lambda)^2] \\
&= \frac{1}{16}[32m^3 + (256 - 96\lambda)m^2 + (-128\lambda - 40\lambda^2)m \\
&\quad - 25\lambda^3 - 40\lambda^2 - 16\lambda] \\
&\geq f(m)/16,
\end{aligned}$$

where

$$f(m) = 32m^3 + 160m^2 - 168m - 81.$$

The sign sequence of the coefficients of $f(m)$ is (+, +, −, −). Using the Descartes' rule of signs, we know that $f(m)$ has a unique positive root. Since $f(6/5)=387/125$, we can judge that the positive root must locate in the interval (0, 6/5). So if Eq.(B3) holds, then $f(m)>0$. Therefore, we obtain $c_2>0$, namely, $b_2>0$ and $a_2=-4\lambda b_2<0$. To sum up, it is obvious that since the sign sequence of the coefficients of $h_1(s)$ is (+, +, ?, −, −, −, −), using the Descartes' rule of sign, we know that $h_1(s)$ has only a unique positive root; that is, if the condition Eq.(B3) holds, then $\phi(t)$ has and only has a zero point in the interval $(0, \pi/2)$. Therefore Lemma 2 is proved.