2

Basic Formulations of the Scaled Boundary Finite Element Method

2.1 Introduction

This chapter presents the basic concepts of the scaled boundary finite element method for two-dimensional stress analysis. The theoretical formulation and implementation are mostly limited to those of simplistic polygonal S-elements (see, for example, Figure 1.3). The most basic 2-node line element (see Figure I.1) is used for the discretization of the edges of an S-element. The scaled boundary finite element formulation for an S-element is developed. From the point of view of development process, this is equivalent to the development of element formulation in the finite element method.

2.2 Modelling of Geometry in Scaled Boundary Coordinates

A key concept of the scaled boundary finite element method is to devise the scaled boundary coordinates to describe the geometry of an S-domain. The role of the scaled boundary coordinates is similar to that of the reference coordinates in the isoparametric finite elements. This coordinate system involves the discretization of the boundary of the S-domain, leading to an S-element. This allows a semi-analytical solution to be obtained.

Same as in the finite element method, the directions of displacements and forces remain in the Cartesian coordinates so that the assembly of S-elements can be conveniently formulated to satisfy the compatibility and the equilibrium.

2.2.1 S-domains: Scaling Requirement on Geometry, Scaling Centre and Scaling of Boundary

In this section, only the geometry of an S-domain is addressed. To guarantee that the so-called scaled boundary transformation introduced in Section 2.2.3 below is unique, the geometry of the S-domain has to meet a so-called *scaling requirement: there exists a region from which every point on the boundary is directly visible, or in other words, there exists a region that has direct line of sight of the whole boundary.* The scaling requirement can also be stated inversely as that there exists a region that is directly visible from any point on the boundary. A point called the *scaling centre* is selected in this region.

The Scaled Boundary Finite Element Method: Introduction to Theory and Implementation,
First Edition. Chongmin Song.

Companion website: www.wiley.com/go/author/scaledboundaryfiniteelementmethod

The scaling requirement on the geometry of a domain is illustrated in Figure 2.1. The domain V shown in Figure 2.1a satisfies the scaling requirement as a point from where every point on the boundary S (the bold solid line) is directly visible can be identified. Such a point is selected as the scaling centre O. The direct lines of sight from the scaling centre to the corners of the domain are drawn as dashed lines. Note that the boundary does not have to be convex and the number of edges is not limited. Figure 2.1b shows an example of a domain that does not satisfy the scaling requirement. No point can be found that has direct visibility of the three points A, B and C on the boundary S at the same time. To apply the scaled boundary finite element method, such a domain has to be subdivided into several smaller ones that satisfy the scaling requirement.

In the scaled boundary finite element method, an S-domain is described by continuously scaling its boundary with respect to a scaling centre. Figure 2.2 depicts the operation to generate the S-domain V by scaling its boundary S. When the boundary is scaled by a factor less than 1, an internal curve similar to the boundary is obtained. Two such internal curves corresponding to scaling factors 0.8 and 0.6, respectively, are shown in Figure 2.2 as thin solid lines. When the scaling is performed continuously from the boundary (with a scaling factor 1) to the scaling centre (with a scaling factor 0), the whole domain is covered. This process is analogous to covering a surface by continuously moving a line. When the scaling requirement is satisfied, a point in the domain is covered once and once only in the scaling process.

The scaling operation to describe an S-domain is addressed in mathematical terms. As shown in Figure 2.3, a system of Cartesian coordinates x, y is defined. Its origin is selected at the scaling centre O. This choice simplifies the derivation of equations, and does not lead to loss of generality. The position vector of a point (x, y) in the S-domain

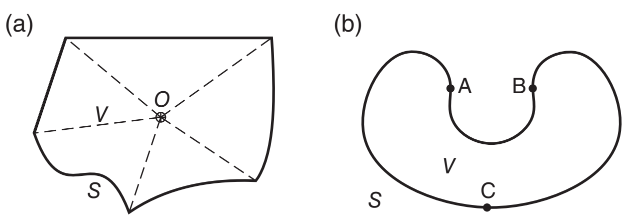


Figure 2.1 Illustration of the scaling requirement. (a) A bounded domain V that satisfies the scaling requirement. The scaling centre is indicated by the marker $\oplus$ from where every point on the boundary S (bold solid line) is directly visible. The dashed lines show examples of direct lines of sight. Such a domain will be referred to as an S-domain. (b) A bounded domain V that does not satisfy the scaling requirement.

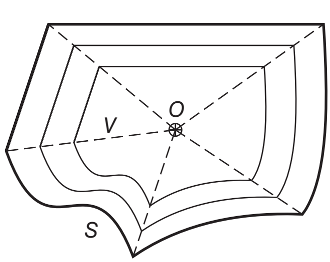


Figure 2.2 Representation of an S-domain V by scaling its boundary S with respect to the scaling centre O selected inside the domain. The dash lines indicate the lines of sight from the scaling centre. The thin lines indicate two typical internal curves resulting from scaling the boundary.

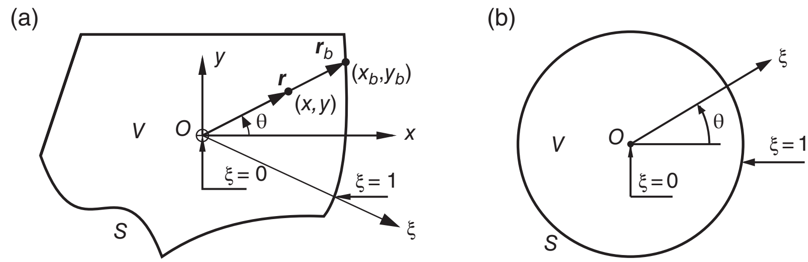


Figure 2.3 (a) Scaling of the boundary of an S-domain using the radial coordinate ξ ($\xi = 1$ at the boundary and $\xi = 0$ at the scaling centre) as the scaling factor. (b) An S-domain is transformed to a unit circular domain in the system of radial coordinate ξ and angular coordinate θ.

(Figure 2.3a) is expressed as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} \tag{2.1}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the x and y directions, respectively. The angular coordinate (i.e. the angle between the $x-$axis and the vector $\mathbf{r}$) is denoted as θ.

Along the radial line connecting the scaling centre and the point (x, y), the point (x_b, y_b) on the boundary is addressed. The point (x, y) and the point (x_b, y_b) have the same angular coordinate θ. The position vector $\mathbf{r}_b$ is expressed as

$$\mathbf{r}_b = x_b\mathbf{i} + y_b\mathbf{j} \tag{2.2}$$

When the scaling requirement is satisfied, no two points on the boundary have the same value of angular coordinate θ. The boundary can be defined by a single-valued function $r_b(\theta)$. It is often more flexible to describe a boundary of complex shape by two single-valued parametric functions $x_b(\theta)$ and $y_b(\theta)$

$$x_b = x_b(\theta) \tag{2.3a}$$

$$y_b = y_b(\theta) \tag{2.3b}$$

where the angular coordinate θ is used as the parametric variable. This parametric description will be used in the scaled boundary coordinates (Section 2.2.3).

A radial coordinate ξ emanating from the scaling centre O is introduced as the scaling factor (Figure 2.3a). The radial coordinate ξ is selected as $\xi = 0$ at the scaling centre and $\xi = 1$ on the boundary. The scaling requirement ensures that the radial coordinate ξ and the angular coordinate θ are not parallel.

As shown in Figure 2.3a, scaling the point (x_b, y_b) to the scaling centre O generates a radial line connecting the two points and passing through the point (x, y). The point (x, y) can thus by obtained by scaling the point (x_b, y_b) on the boundary with the radial coordinate ξ as

$$\mathbf{r} = \xi\mathbf{r}_b \tag{2.4}$$

The scaling operation is expressed in the Cartesian coordinates as

$$x = \xi x_b \tag{2.5a}$$

$$y = \xi y_b \tag{2.5b}$$

Equation (2.5) can be regarded as a coordinate transformation between (x, y) and (ξ, θ). As shown in Figure 2.3b, an S-domain is transformed into a unit circular domain in the system of radial coordinate ξ and angular coordinate θ. The boundary is specified by a constant radial coordinate $\xi = 1$. This choice simplifies the enforcement of boundary conditions in solving differential equations, much like the use of polar coordinates in the solution of problems on circular domains. In fact, the coordinate system $(\xi,\ \theta)$ resembles the polar coordinates r and θ. Using Eq. (2.5), the radial coordinate r is written as

$$r = \xi r_b = \xi \sqrt{x_b^2 + y_b^2} \tag{2.6a}$$

where

$$r_b = \sqrt{x_b^2 + y_b^2} \tag{2.6b}$$

is the radial coordinate on boundary. The angular coordinate θ is expressed as

$$\theta = \arctan \frac{y_b}{x_b} \tag{2.6c}$$

with the principal value $-\pi < \theta \leq \pi$. The scaling requirement guarantees that the angle θ is a unique valued function along the boundary.

It is worthwhile noting the following remarks on the scaling requirement and the S-domain.

1) Star-shaped domains in mathematics (Stewart and Tall, 1983) are S-domains. A domain is star-shaped if there exists a point such that a straight line segment connecting this point with any other point in the domain is contained in the domain. In two and three dimensions, this criterion is equivalent to having direct lines of sight of the whole boundary from a point. Figure 2.4a depicts an example of star-shaped polygonal domains in computational geometry (Preparata and Shamos, 1985). The lines of sight are shown by dashed lines. The set of points that have direct line of sight of the whole boundary is called the kernel of the star-shaped polygon domain as shown by the shaded area in Figure 2.4b. Therefore, a star-shaped domain meets the scaling requirement and the scaling centre is selected within the kernel. A star-shaped domain is not necessarily convex. As shown in Figure 2.4, some parts of the boundary can be concave. All convex polygons are star-shaped. A convex

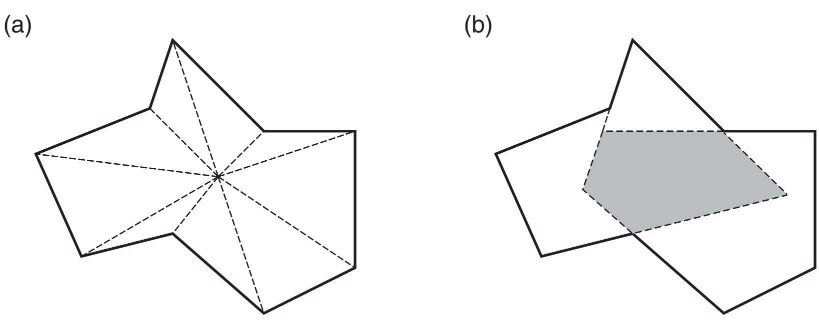


Figure 2.4 Star-shaped polygonal domain. (a) The dashed lines are the straight line segments that connect a point with the vortices of the polygonal domain. (b) The shaded region shows the kernel of the star-shaped polygonal domain.

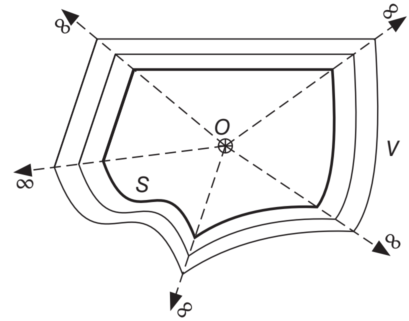


Figure 2.5 An unbounded S-domain. The unbounded domain V is covered by scaling its boundary S with respect to the scaling centre O selected outside of the domain. The thin lines indicate internal curves resulting from scaling the boundary with a factor larger than 1. The dashed lines indicate the lines of sight from the scaling centre.

polygon coincides with its own kernel, i.e. the whole boundary of a convex polygon is directly visible from any point in the polygon.

On the other hand, an S-domain may not be a star-shaped domain. Some of these cases are given below.

2) The scaled boundary technique is also applicable to the modelling of unbounded (or exterior) domains. An unbounded domain is depicted in Figure 2.5. The scaling centre O is selected outside of the unbounded domain. Since the whole boundary is directly visible from the scaling centre, the unbounded domain is an S-domain, but not a star-shaped domain. The unbounded S-domain V is represented by scaling its boundary S with a scaling factor varying continuously from 1 at the boundary to infinity. The two internal curves obtained by scaling the boundary with the scaling factors 1.1 and 1.2, respectively, are shown in Figure 2.5 as thin solid lines.
3) Two useful extensions to the above base cases of forming S-domains by scaling their boundaries are described briefly in the following and will be addressed in detail in the subsequent chapters.
   a) The first is the modelling of problems with a straight edge crack and material interfaces as illustrated in Figure 2.6. Figure 2.6a is an edge-cracked square. The scaling centre is selected at the crack tip. The two straight crack faces passing through the scaling centre are not included in the scaling process and are denoted as 'side-face'. The rest of the boundary is referred to as the 'defining curve', which is not closed. The S-domain of the cracked square is obtained by continuously scaling the defining curve with respect to the scaling centre. The side-faces (crack faces) are formed in the scaling process by the two ends of the defining curve.

      A corner formed by three materials occupying the domains V_1, V_2 and V_3 is shown in Figure 2.6b. The scaling centre is selected at the vertex. The defining curves for the three domains are plotted by bold solid lines. They are jointed at their common ends denoted by circles. The three domains are obtained by continuously scaling the defining curves and are modelled as one S-domain. The side-faces and material interfaces are formed in the scaling process by the ends of the defining curves.

      As will be demonstrated in Example 3.9 and in Chapter 10, the scaled boundary finite element method leads to a semi-analytical solution of the strain/stress singularity at the crack tip and multi-material corners. Accurate results can be obtained without a local mesh refinement or special element, which represents

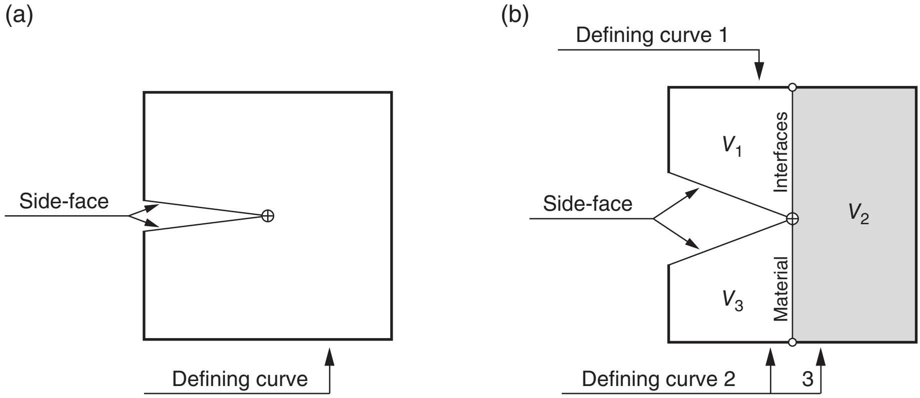


Figure 2.6 (a) Modelling of an edge-cracked square by scaling the defining curve. The two side-faces (crack faces) are formed by scaling the two ends of the defining curve. (b) Modelling of a three-material corner by scaling the defining curves. The two side-faces and the material interfaces are formed by scaling the ends of the defining curves.

one of the advantages of the scaled boundary finite element method in computational fracture mechanics.

b) The second extension is the modelling of a half-plane with an excavation as shown in Figure 2.7. The problem domain is modelled as one S-domain. The scaling centre is selected outside of the domain and on the extension of the free surface of the half-plane. The free surface is not included in the scaling process and denoted as 'side-face'. The excavation line is referred to as the defining curve and scaled with respect to the scaling centre. The S-domain with the two side-faces representing the free surface are formed by scaling the defining curve. A semi-analytical solution satisfying the boundary conditions at infinity can be obtained without discretizing the half-plane.

4) The visibility of the boundary from the scaling centre is determined by the angle between the boundary and the line of sight from the scaling centre. When the angle is a right angle, the visibility is the highest. When the angle is very small (or very close to 180°) like those ones shaded in Figure 2.8a, the low visibility of the boundary from the scaling centre will affect the numerical accuracy of the scaled boundary transformation and the scaled boundary finite element analysis. This is analogous to the effect of elements with small internal angles on the accuracy of a finite element analysis. To improve the visibility of the boundary, i.e. to avoid very small angles between

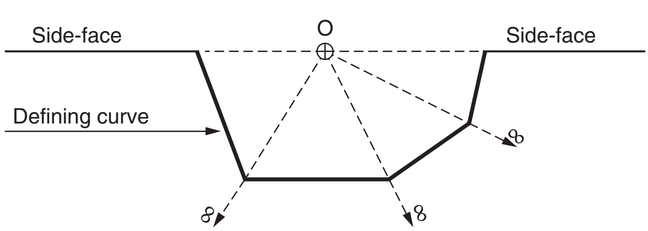


Figure 2.7 Modelling of an excavation in a half-plane as an S-domain by scaling the defining curve. The two side-faces (free surface) are formed by scaling the two ends of the defining curve.

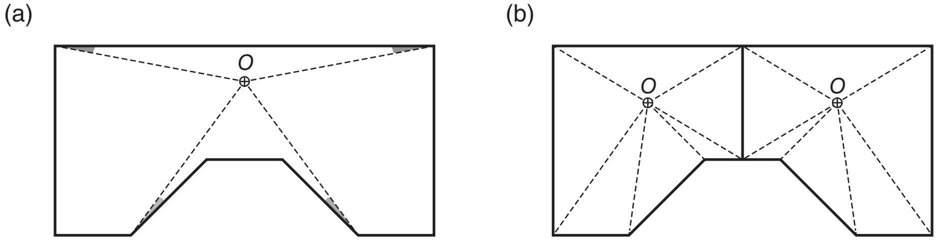


Figure 2.8 (a) An S-domain with small angles between the boundary and the lines of sight from the scaling centre, as indicated by the shaded areas, leading to low visibility of the boundary. (b) Subdivision of the S-domain to increase angles between the boundary and the lines of sight from the scaling centre and, thus, the visibility of boundary.

the lines of sight and the boundary, an S-domain can be subdivided as shown in Figure 2.8b.

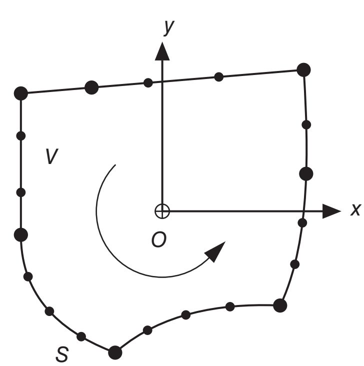
2.2.2 S-elements: Boundary Discretization of S-domains

In practical applications of the scaled boundary finite element method, the boundary of an S-domain can be any general shape satisfying the scaling requirement. Generally speaking, the solution of a problem is only feasible in semi-analytical form with a piece-wise discretized description of the boundary. As shown in Figure 2.9, the boundary S of the S-domain V is divided into line elements. The large dots indicate the end nodes of elements. The reference coordinate of the line elements follows the counter-clockwise direction around the scaling centre (i.e. the right-hand rule). The S-domain and its boundary discretization define an S-element. The solution inside the S-element will be obtained analytically, leading to a semi-analytical procedure.

Standard isoparametric formulations of the finite element method apply to describe the geometry of the elements on the boundary. The Cartesian coordinates of the nodes of an element on boundary are arranged in $\{x\}$, $\{y\}$. The geometry and displacements of the isoparametric line element are interpolated using the same shape functions from their nodal values.

A 2-node element is shown in Figure 2.10a. It is the simplest element and will be used in all the examples in this chapter. The nodal coordinate vectors of the element

Figure 2.9 An S-element obtained by discretizing the boundary S of S-domain V. Displacement-based elements of different orders are used. The direction of the elements has to follow the counter-clockwise direction around the scaling centre.



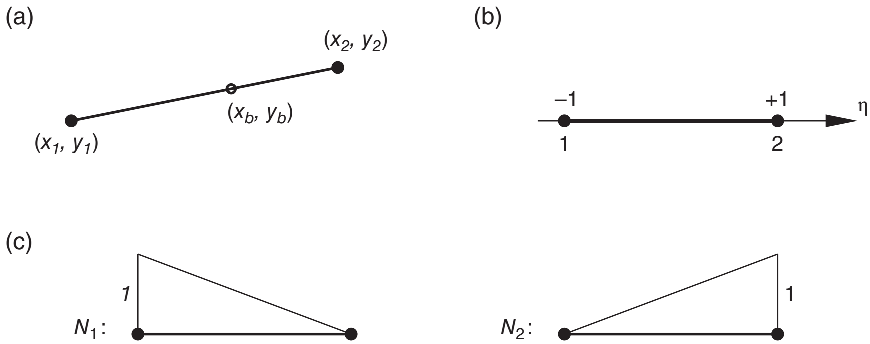


Figure 2.10 Two-node line element on boundary. The element direction must follow the counter-clockwise direction around the scaling centre. (a) Physical element. A point $(x_b,\ y_b)$ on the element is obtained by interpolating nodal coordinates, see Eq. (2.17). (b) Parent element in natural coordinate η. (c) Shape functions in natural coordinate η.

are equal to

$$\{x\} = [\, x_1 \;\; x_2 \,]^T \tag{2.7a}$$

$$\{y\} = [\, y_1 \;\; y_2 \,]^T \tag{2.7b}$$

The coordinates of an arbitrary point on the element (on the boundary) are denoted as x_b, y_b.

The parent element in the natural coordinate η (ξ has been used for the radial coordinate) is illustrated in Figure 2.10b. It has a length of 2 units. The natural coordinates of nodes 1 and 2 are equal to -1 and $+1$, respectively. The linear interpolation of, for example, the x–coordinate on the 2-node element is expressed as

$$x_b(\eta) = a_0 + a_1\eta \tag{2.8}$$

The interpolation constants a_0 and a_1 are determined by formulating Eq. (2.8) at the two nodes, leading to

$$x_1 = x_b(-1) = a_0 + a_1 \times (-1) \tag{2.9a}$$

$$x_2 = x_b(+1) = a_0 + a_1 \times (+1) \tag{2.9b}$$

The solution of Eq. (2.9) is expressed as

$$a_0 = \frac{1}{2}(x_1 + x_2) \tag{2.10a}$$

$$a_1 = \frac{1}{2}(x_2 - x_1) \tag{2.10b}$$

Using Eq. (2.10), Eq. (2.8) is written as

$$x_b(\eta) = \frac{1}{2}(x_1 + x_2) + \frac{1}{2}(x_2 - x_1)\eta = \frac{1}{2}(1 - \eta)x_1 + \frac{1}{2}(1 + \eta)x_2 \tag{2.11}$$

The interpolation in Eq. 2.11 is expressed in terms of the nodal coordinates as

$$x_b(\eta) = N_1(\eta)x_1 + N_2(\eta)x_2 \tag{2.12}$$

where the shape functions are equal to

$$N_1(\eta) = \frac{1}{2}(1 - \eta) \tag{2.13a}$$

$$N_2(\eta) = \frac{1}{2}(1 + \eta) \tag{2.13b}$$

The two shape functions $N_1(\eta)$ and $N_2(\eta)$ are plotted in Figure 2.10c. It is easy to verify that the shape functions possess the following properties:

- Kronecker delta functions

$$N_i(\eta_j) = \delta_{ij} = \begin{cases} 1 & \text{when} \quad i = j \\ 0 & \text{when} \quad i \neq j \end{cases} \tag{2.14}$$

- Partition of unity

$$\sum_i N_i(\eta) = 1 \tag{2.15}$$

The shape functions are expressed in matrix form as

$$[N(\eta)] = \left[\frac{1}{2}(1 - \eta) \;\; \frac{1}{2}(1 + \eta) \right] \tag{2.16}$$

The interpolation of coordinate x_b in Eq. (2.12) and, similarly, the interpolation of coordinate y_b are written as

$$x_b = [N]\{x\} \tag{2.17a}$$

$$y_b = [N]\{y\} \tag{2.17b}$$

where, for conciseness, the arguments η have been omitted from the coordinates $x_b = x_b(\eta)$, $y_b = x_b(\eta)$ and the shape functions $[N] = [N(\eta)]$.

A 3-node line element is shown Figure 2.11a. The nodal coordinate vectors are expressed as

$$\{x\} = [\, x_1 \;\; x_2 \;\; x_3 \,]^T \tag{2.18a}$$

$$\{y\} = [\, y_1 \;\; y_2 \;\; y_3 \,]^T \tag{2.18b}$$

The parent element in the natural coordinate η is shown in Figure 2.11b. The shape functions are equal to

$$[N(\eta)] = \left[\frac{1}{-2}\eta(1 - \eta) \;\; 1 - \eta^2 \;\; \frac{1}{2}\eta(1 + \eta) \right] \tag{2.19}$$

and plotted in Figure 2.10c. They have the properties of the Kronecker delta and satisfy the partition of unity.

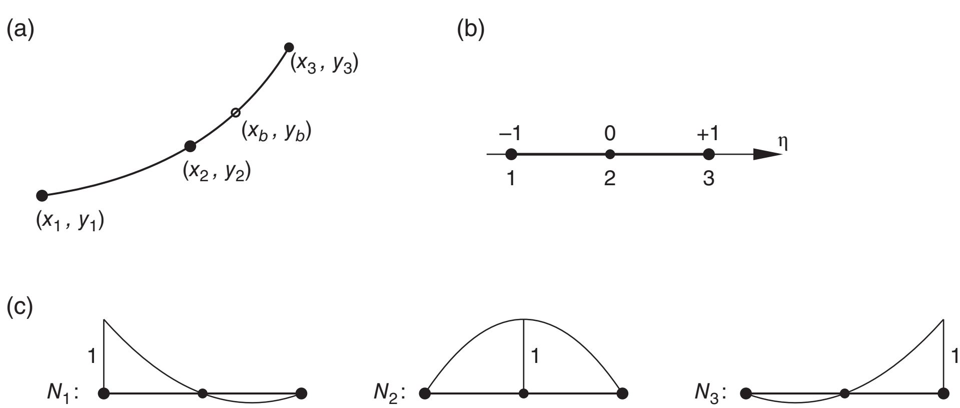


Figure 2.11 Three-node line element on boundary. The element direction must follow the counter-clockwise direction around the scaling centre. (a) Physical element. A point $(x_b,\ y_b)$ on the element is obtained by interpolating nodal coordinates, see Eq. (2.17). (b) Parent element in natural coordinate η. (c) Shape functions in natural coordinate η.

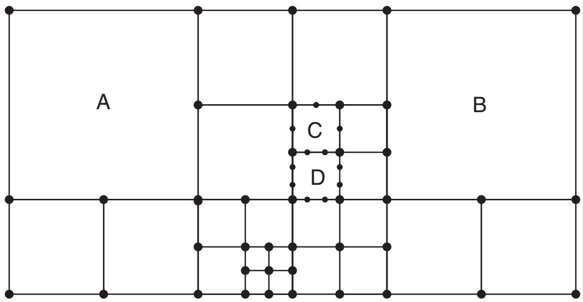


Figure 2.12 A quadtree mesh illustrating the simplicity of mesh generation and remeshing using S-elements. This mesh of S-elements satisfies displacement compatibility requirement.

The only requirement for the boundary discretization as shown in Figure 2.9 is that the displacement is continuous at the end nodes where the elements are connected. As long as this requirement is satisfied, the elements can be of any type and order. They can also be mixed together in one S-element. In addition, one edge of an S-element can have more than one element As one edge is shared by two S-elements only, compatibility is automatically satisfied. Practically, any displacement-based elements can be used.

The fact that an S-element allows a flexible boundary discretization reduces significantly the burden on conventional finite element mesh generation and remeshing. This is demonstrated by the simple quadtree mesh shown in Figure 2.12. A quadtree mesh is highly efficient and suitable for adaptive analysis, but its application in finite element analysis is greatly hindered by the presence of hanging nodes causing displacement incompatibility (Zienkiewicz et al., 2005). When a quadtree cell is modelled as one S-element in the scaled boundary finite element method, this difficulty is avoided by subdividing an edge at a hanging node into two or more line elements. For example, the lower and right edges of S-element A in Figure 2.12 are divided into two 2-node line elements so that the compatibility with the adjacent S-elements of a smaller size is satisfied. Similarly, the left edge of S-element B is divided into three 2-node line elements by the two hanging nodes. S-elements C and D depict the mesh refinement by increasing the element order (p-refinement). The compatibility with adjacent S-elements is satisfied automatically on the common edges. The refinement of one S-element will affect at most the four adjacent S-elements.

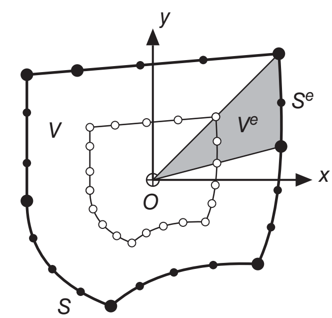
2.2.3 Scaled Boundary Transformation

As explained in Section 2.2.1, the S-domain is described by scaling the boundary. For the discretized boundary of an S-element, the scaling operation applies to individual elements. As shown in Figure 2.13, the line element S^e covers a sector V^e of the S-element when it is scaled towards the scaling centre. The S-element is the assembly of all the sectors resulting from scaling the individual elements on the boundary. Therefore, one sector covered by the scaling of a single line element is addressed in the derivation of the scaled boundary finite element equation. The equations for the S-element are obtained by enforcing the compatibility and equilibrium between the sectors.

2.2.3.1 Scaled Boundary Coordinates

The boundary scaling discussed in Section 2.2.1 is applied to a single line element on the boundary. The process is depicted in Figure 2.14 using a 3-node element. Scaling the point (x_b, y_b) on the line element leads to a radial line. The coordinates of a point (x, y)

Figure 2.13 Representation of an S-element by scaling the line elements on boundary.



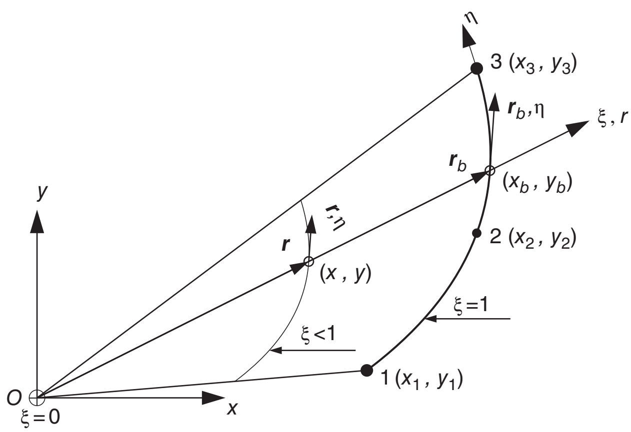


Figure 2.14 Scaled boundary coordinates defined by scaling a line element at the boundary.

along the radial line and inside the domain are obtained by substituting Eq. (2.17) into Eq. (2.5)

$$x = \xi[N(\eta)]\{x\} \tag{2.20a}$$

$$y = \xi[N(\eta)]\{y\} \tag{2.20b}$$

ξ, η are called *the scaled boundary coordinates* in two dimensions. ξ is a (dimensionless) radial coordinate. η is the circumferential coordinate. They form a right-hand coordinate system. Equation (2.20) defines the *scaled boundary transformation*, which transforms the coordinates between x, y and ξ, η. The scaling requirement ensures the uniqueness of the transformation.

The scaled boundary transformation can be regarded as a semi-analytical approach to transform an S-element into a circular domain. For example, the boundary S of the S-element V in Figure 2.13 is transformed into a circle described by a constant radial coordinate $\xi = 1$ as illustrated in Figure 2.15 (see also Figure 2.3). The S-element is specified by $0 \leq \xi \leq 1$.

The circumferential direction, parallel to the boundary, around the scaling centre O is described by the element number of the scaled element and the local coordinate η as in one-dimensional finite elements. A straight line passing through the scaling centre O in x, y coordinates remains as a straight line and is described by a constant η. The angular

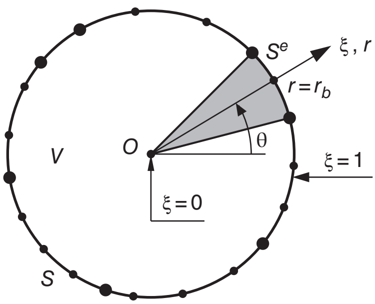


Figure 2.15 Representation of the polygonal domain shown in Figure 2.13 in the scaled boundary coordinates.

coordinate θ in Eq. (2.6c) is expressed as a function of the circumferential coordinate η

$$\theta(\eta) = \arctan \frac{y_b(\eta)}{x_b(\eta)} \tag{2.21}$$

which is independent of the radial coordinate ξ. Since every point on the boundary of an S-domain is directly visible, the function $\theta(\eta)$ is single-valued. The element number and the circumferential coordinate η of the element can be regarded as a discrete representation of the angular coordinate θ. An element on the boundary in Figure 2.13 is transformed into an arc defined by $\xi = 1$ and $-1 \leq \eta \leq 1$ in Figure 2.15. The part of the domain covered by scaling an element on the boundary in Figure 2.13 is a sector defined by $0 \leq \xi \leq 1$ and $-1 \leq \eta \leq 1$ in Figure 2.15.

2.2.3.2 Coordinate Transformation of Partial Derivatives

The scaled boundary transformation in Eq. (2.20) is similar to the coordinate transformation in isoparametric finite elements (Cook et al., 2002; Zienkiewicz et al., 2005). The function $\xi[N(\eta)]$ is equivalent to the shape functions of an isoparametric finite element from the viewpoint of coordinate transformation. The procedure for coordinate transformation in the formulation of isoparametric finite elements is followed to perform the scaled boundary transformation.

The transformation of partial derivatives of the spatial dimensions is addressed. In the governing differential equations (Section A.2 on page 454), the displacement field is expressed in the Cartesian coordinates x and y. The partial derivatives with respect to x and y are required. In the scaled boundary finite element method (as in the isoparametric finite element formulation), the displacement field is written in the scaled boundary coordinates ξ and η (see Eq. (2.78) in Section 2.4). The partial derivatives with respect to the Cartesian coordinates x and y are not available explicitly and are obtained from those with respect to the scaled boundary coordinates ξ and η.

Applying the chain rule, the partial differential operators with respect to the scaled boundary coordinates ξ and η are expressed as[1]

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} \tag{2.22a}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} \tag{2.22b}$$

1 More details of the derivation of the transformation of spatial derivative are presented in Section 6.4 for the three-dimensional case.

It is expressed in the matrix form as

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \tag{2.23}$$

with the Jacobian matrix defined as

$$[J] = \begin{bmatrix} x,_{\xi} & y,_{\xi} \\ x,_{\eta} & y,_{\eta} \end{bmatrix} \tag{2.24}$$

A comma followed by a subscript is used to denote partial differentiation with respect to the variable in the subscript. The derivatives with respects to x, y are transformed into those with respect to ξ, η by inverting Eq. (2.23), resulting in

$$\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} \tag{2.25}$$

The partial derivatives in the Jacobian matrix (Eq. (2.24)) are obtained from Eq. (2.20) (or equivalently Eqs. (2.5) and (2.17)) as

$$x,_{\xi} = x_b = [N]\{x\} \tag{2.26a}$$

$$x,_{\eta} = \xi x_{b,\eta} = \xi [N],_{\eta} \{x\} \tag{2.26b}$$

$$y,_{\xi} = y_b = [N]\{y\} \tag{2.26c}$$

$$y,_{\eta} = \xi y_{b,\eta} = \xi [N],_{\eta} \{y\} \tag{2.26d}$$

Substituting Eq. (2.26) into Eq. (2.24) and separating the coordinates ξ from η yields

$$[J] = \mathrm{diag}(1, \ \xi)[J_b] \tag{2.27}$$

where $[J_b]$ is the Jacobian matrix at the boundary ($\xi = 1$)

$$[J_b] = \begin{bmatrix} x_b & y_b \\ x_{b,\eta} & y_{b,\eta} \end{bmatrix} \tag{2.28}$$

It is a function of η and depends on the geometry of the element only. Substituting Eq. (2.27) into Eq. (2.25) results in

$$\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = [J_b]^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{1}{\xi}\dfrac{\partial}{\partial \eta} \end{Bmatrix} \tag{2.29}$$

The inverse of the Jacobian at the boundary $[J_b]$ is written as

$$[J_b]^{-1} = \frac{1}{|J_b|} \begin{bmatrix} y_{b,\eta} & -y_b \\ -x_{b,\eta} & x_b \end{bmatrix} \tag{2.30}$$

The determinant of $[J_b]$ is expressed as

$$|J_b| = x_b y_{b,\eta} - y_b x_{b,\eta} \tag{2.31}$$

Substituting Eq. (2.30) into Eq. (2.29) results in the transformation of the differential operators

$$\left\{\begin{array}{c}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\end{array}\right\} = \frac{1}{|J_b|}\left\{\begin{array}{c} y_{b,\eta} \\ -x_{b,\eta}\end{array}\right\}\frac{\partial}{\partial \xi} + \frac{1}{|J_b|}\frac{1}{\xi}\left\{\begin{array}{c} -y_b \\ x_b\end{array}\right\}\frac{\partial}{\partial \eta} \tag{2.32}$$

The partial derivatives with respect to ξ and η are separated to different terms.

2.2.3.3 Geometrical Properties in Scaled Boundary Coordinates

The geometrical properties of an S-element, which are required for the derivation of the scaled boundary finite element equation in the subsequent sections, are formulated in the scaled boundary coordinates.

The boundary of the S-element is considered. The part of the boundary represented by a line element is depicted in Figure 2.14. The position vector of a point (x_b, y_b) is given in Eq. (2.2). Its tangential vector is expressed as

$$\mathbf{r}_{b,\eta} = x_{b,\eta}\mathbf{i} + y_{b,\eta}\mathbf{j} \tag{2.33}$$

It is shown that the determinant of the Jacobian on boundary, $|J_b|$ in Eq. (2.31), is equal to the area of the parallelogram formed by vectors $\mathbf{r}_b$ and $\mathbf{r}_{b,\eta}$

$$\begin{aligned} |\mathbf{r} \times \mathbf{r},_\eta| &= \begin{vmatrix} 1 & 1 & \mathbf{k} \\ x_b & y_b & 0 \\ x_{b,\eta} & y_{b,\eta} & 0 \end{vmatrix} \\ &= x_b y_{b,\eta} - y_b x_{b,\eta} \\ &= |J_b| \end{aligned} \tag{2.34}$$

where $\mathbf{k}$ is the unit vector along z (perpendicular to $x - y$ plane) direction. The vector $\mathbf{r}_b$ represents the line of sight from the scaling centre to the point (x_b, y_b) and the vector $\mathbf{r}_{b,\eta}$ the tangential direction of the boundary. When the scaling requirement is met, the area of the parallelogram will not be equal to 0. Since the scaled boundary coordinate system follows the right-hand rule, $|J_b|$ is always positive and the Jacobian matrix $[J_b]$ is invertible. The scaled boundary transformation is thus well defined. In a scaled boundary finite element analysis, the angle between the vector $\mathbf{r}_b$ and vector $\mathbf{r}_{b,\eta}$ should not be too small (Figure 2.8) so that the determinant $|J_b|$ is not close to 0 and the Jacobian matrix $[J_b]$ is well-conditioned.

A sector covered by scaling an element at the boundary as depicted in Figure 2.16 is considered. Several coordinate curves with constant values of scaled boundary coordinates ξ and η are plotted in Figure 2.16. Those with constant values of the radial coordinate ξ, denoted as S_r, are shown as solid lines, and these with constant values of the circumferential coordinate η, denoted as S_c, are shown as dashed lines.

A coordinate curve S_c with a constant ξ is considered. Its tangential vector $\mathbf{r},_\eta$, shown in Figure 2.16, is equal to the derivative of the position vector (Eq. (2.1)) with respect to η. Using Eq. (2.4), it is written as

$$\mathbf{r},_\eta = \xi\mathbf{r}_{b,\eta} \tag{2.35}$$

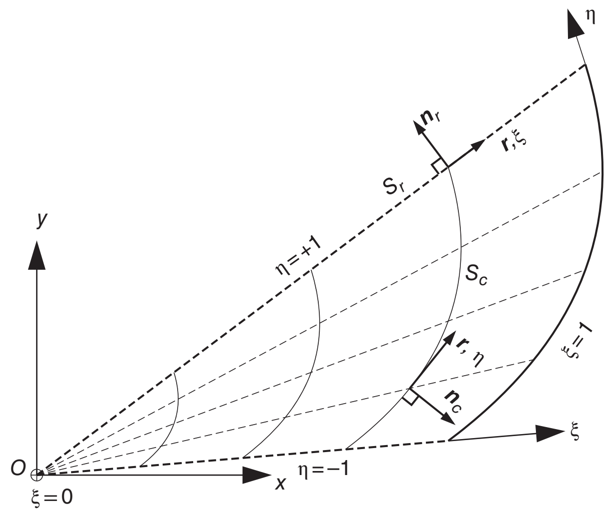


Figure 2.16 A sector of an S-element covered by scaling an element at the boundary.

where $\mathbf{r}_{b,\eta}$ (Eq. (2.33)) is the tangential vector at the boundary ($\xi = 1$). The magnitude of the tangential vector $\mathbf{r},_\eta$ (Eq. (2.35)) is expressed as

$$|\mathbf{r},_\eta| = \xi|\mathbf{r}_{b,\eta}| \tag{2.36}$$

where the magnitude of the tangential vector at the boundary $|\mathbf{r}_{b,\eta}|$ (Eq. (2.33)) is equal to

$$|\mathbf{r}_{b,\eta}| = \sqrt{\left(x_{b,\eta}\right)^2 + \left(y_{b,\eta}\right)^2} \tag{2.37}$$

An infinitesimal length on S_c is equal to (Eq. (2.36))

$$\mathrm{d}S_c = |\mathbf{r},_\eta|\mathrm{d}\eta = \xi|\mathbf{r}_{b,\eta}|\mathrm{d}\eta \tag{2.38}$$

Substituting Eq. (2.33) into Eq. (2.35) results in

$$\mathbf{r},_\eta = \xi x_{b,\eta}\mathbf{i} + \xi y_{b,\eta}\mathbf{j} \tag{2.39}$$

which can also be obtained directly by differentiating Eq. (2.1) with respect to η and using Eq. (2.5). The unit outward normal vector $\mathbf{n}_c$, shown in Figure 2.16, and the tangential vector of a coordinate curve S_c form a right-hand coordinate system. The unit outward normal vector is obtained using Eq. (2.39) as

$$\mathbf{n}_c = \frac{y_{b,\eta}}{|\mathbf{r}_{b,\eta}|}\mathbf{i} - \frac{x_{b,\eta}}{|\mathbf{r}_{b,\eta}|}\mathbf{j} \tag{2.40}$$

It is independent of ξ, i.e., the unit outward normal vector at a given η is the same for any coordinate curve S_c. For later use, the unit outward normal vector $\mathbf{n}_c$ is expressed in matrix form as

$$\{n_c\} = \begin{Bmatrix} n_{cx} \\ n_{cy} \end{Bmatrix} = \frac{1}{|\mathbf{r}_{b,\eta}|}\begin{Bmatrix} y_{b,\eta} \\ -x_{b,\eta} \end{Bmatrix} \tag{2.41}$$

A coordinate curve S_r with a constant η is a radial line passing through the scaling centre (Figure 2.16). Its tangential (direction) vector, $\mathbf{r}_{,\xi}$, is obtained from Eq. (2.4) as

$$\mathbf{r}_{,\xi} = \mathbf{r}_b \tag{2.42}$$

with the position vector $\mathbf{r}_b$ given in Eq. (2.2). It is expressed in the Cartesian coordinate system as

$$\mathbf{r}_{,\xi} = x_b\mathbf{i} + y_b\mathbf{j} \tag{2.43}$$

Its unit outward normal vector $\mathbf{n}_r$ (Figure 2.16) is expressed as

$$\mathbf{n}_r = -\frac{y_b}{r_b}\mathbf{i} + \frac{x_b}{r_b}\mathbf{j} \tag{2.44}$$

where r_b is the magnitude of the position vector $\mathbf{r}_b$ (Eq. (2.6b)). For later use, the unit outward normal vector $\mathbf{n}_r$ is written in matrix form as

$$\{n_r\} = \begin{Bmatrix} n_{rx} \\ n_{ry} \end{Bmatrix} = \frac{1}{r_b}\begin{Bmatrix} -y_b \\ x_b \end{Bmatrix} \tag{2.45}$$

For infinitesimal changes dξ, dη of the scaled boundary coordinates, the signed infinitesimal volume dV is equal to the magnitude of the cross product of the infinitesimal increments of the tangential vectors (Eqs. (2.39) and (2.43))

$$\begin{aligned} \mathrm{d}V &= |(\mathbf{r}_{,\xi}\,\mathrm{d}\xi) \times (\mathbf{r}_{,\eta}\,\mathrm{d}\eta)| \\ &= \begin{vmatrix} 1 & 1 & \mathbf{k} \\ x_b & y_b & 0 \\ \xi x_{b,\eta} & \xi y_{b,\eta} & 0 \end{vmatrix} \mathrm{d}\xi\mathrm{d}\eta \\ &= \xi(x_b y_{b,\eta} - y_b x_{b,\eta})\mathrm{d}\xi\mathrm{d}\eta \end{aligned} \tag{2.46}$$

Using Eq. (2.31), Eq. (2.46) is expressed as

$$\mathrm{d}V = \xi|J_b|\mathrm{d}\xi\mathrm{d}\eta \tag{2.47}$$

Note that $|J_b|$ is positive.

Example 2.1 Two-node Element: Scaled Boundary Transformation

A 2-node line element with the nodal coordinates (x_1, y_1) and (x_2, y_2) is shown in Figure 2.17. Perform the scaled boundary transformation of sector covered by scaling this element.

1) Boundary discretization
   Note that the nodes are numbered in such a manner that the natural coordinate η of the element (see Figure 2.17) follows the counter-clockwise direction around the scaling centre O. The nodal coordinates are arranged as

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{2.48a}$$

$$\{y\} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} \tag{2.48b}$$

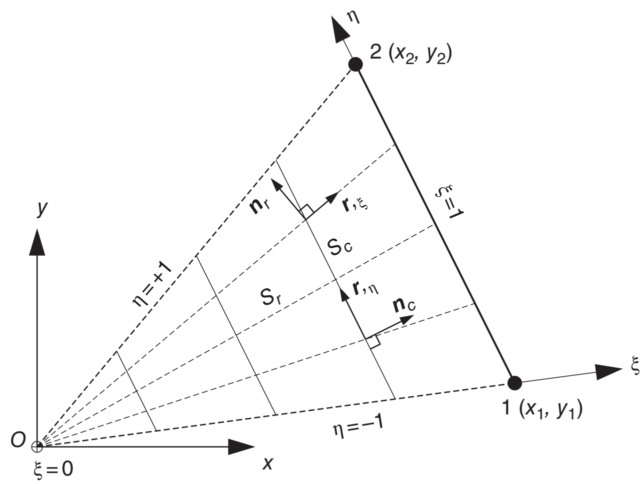


Figure 2.17 Scaled boundary coordinate transformation of a 2-node line element.

Substituting the nodal coordinate vectors $\{x\}$, $\{y\}$ and the shape functions in Eq. (2.16) into Eq. (2.17), the geometry of the line element is given in Cartesian coordinates by

$$x_b = [N]\{x\} = \frac{1}{2}(1-\eta)x_1 + \frac{1}{2}(1+\eta)x_2 = \bar{x} + \frac{1}{2}\Delta_x\eta \tag{2.49a}$$

$$y_b = [N]\{y\} = \frac{1}{2}(1-\eta)y_1 + \frac{1}{2}(1+\eta)y_2 = \bar{y} + \frac{1}{2}\Delta_y\eta \tag{2.49b}$$

with the abbreviations

$$\Delta_x = x_2 - x_1 \tag{2.50a}$$

$$\Delta_y = y_2 - y_1 \tag{2.50b}$$

and

$$\bar{x} = \frac{1}{2}\left(x_1 + x_2\right) \tag{2.51a}$$

$$\bar{y} = \frac{1}{2}\left(y_1 + y_2\right) \tag{2.51b}$$

For later use, differentiating Eq. (2.49) leads to

$$x_{b,\eta} = \frac{1}{2}\Delta_x \tag{2.52a}$$

$$y_{b,\eta} = \frac{1}{2}\Delta_y \tag{2.52b}$$

2) Scaled boundary coordinates
Substituting Eq. (2.49) into Eq. (2.5) results in

$$x = \xi x_b = \xi\left(\bar{x} + \frac{1}{2}\Delta_x\eta\right) \tag{2.53a}$$

$$y = \xi y_b = \xi\left(\bar{y} + \frac{1}{2}\Delta_y\eta\right) \tag{2.53b}$$

This equation defines the transformation between the Cartesian coordinates x, y and the scaled boundary coordinates ξ, η. As shown in Figure 2.17, coordinate curves S_c with constant ξ are parallel to the line element. Coordinate curves S_r with constant η are radial lines connecting the scaling centre O and the points on the line element.

3) Coordinate transformation
To obtain the Jacobian matrix defined in Eq. (2.24), the partial derivatives of x, y with respect to ξ, η are evaluated from Eq. (2.53)

$$x,_{\xi} = x_b = \bar{x} + \frac{1}{2}\Delta_x\eta \tag{2.54a}$$

$$y,_{\xi} = y_b = \bar{y} + \frac{1}{2}\Delta_y\eta \tag{2.54b}$$

$$x,_{\eta} = \xi x_{b,\eta} = \frac{1}{2}\Delta_x\xi \tag{2.54c}$$

$$y,_{\eta} = \xi y_{b,\eta} = \frac{1}{2}\Delta_y\xi \tag{2.54d}$$

leading to

$$[J] = \text{diag}(1,\ \xi)[J_b] \tag{2.55}$$

where the Jacobian matrix at the boundary is expressed as

$$[J_b] = \begin{bmatrix} \bar{x} + \frac{1}{2}\Delta_x\eta & \bar{y} + \frac{1}{2}\Delta_y\eta \\ \frac{1}{2}\Delta_x & \frac{1}{2}\Delta_y \end{bmatrix} \tag{2.56}$$

The same result can also be obtained by directly substituting Eq. (2.49) and Eq. (2.52) into Eq. (2.28). The determinant of the Jacobian matrix on the boundary (Eq. (2.31)) is expressed, after subtracting the second row multiplied with η from the first row,as

$$|J_b| = \frac{1}{2}\begin{vmatrix} \bar{x} & \bar{y} \\ \Delta_x & \Delta_y \end{vmatrix} = \frac{1}{2}\left(\bar{x}\Delta_y - y\bar{\Delta}_x\right) \tag{2.57}$$

Using Eqs. (2.50) and (2.51), Eq. (2.57) is simplified as

$$|J_b| = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \frac{1}{2}(x_1y_2 - x_2y_1) \tag{2.58}$$

$|J_b|$ is equal to the area of the triangle covered by scaling the 2-node element (Figure 2.17). To ensure that the Jacobian matrix is well conditioned, small angles between the boundary and the line of sight from the scaling centre are to be avoided (Figure 2.8).
The inverse of the Jacobian $[J_b]$ (Eq. (2.30)) is expressed as

$$[J_b]^{-1} = \frac{1}{|J_b|}\begin{bmatrix} \frac{1}{2}\Delta_y & -\left(\bar{y} + \frac{1}{2}\Delta_y\eta\right) \\ \frac{1}{2}\Delta_x & \bar{x} + \frac{1}{2}\Delta_x\eta \end{bmatrix} \tag{2.59}$$

It provides the transformation of the partial derivatives in the Cartesian coordinates to those in the scaled boundary coordinates (Eq. (2.32))

$$\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = \frac{1}{|J_b|} \begin{Bmatrix} \dfrac{1}{2}\Delta_y \\ \dfrac{1}{2}\Delta_x \end{Bmatrix} \frac{\partial}{\partial \xi} + \frac{1}{|J_b|}\frac{1}{\xi} \begin{Bmatrix} -\left(\bar{y} + \dfrac{1}{2}\Delta_y \eta\right) \\ \bar{x} + \dfrac{1}{2}\Delta_x \eta \end{Bmatrix} \frac{\partial}{\partial \eta} \tag{2.60}$$

4) Geometry properties of coordinate transformation
The tangential vector of a coordinate curve S_c with a constant ξ (Figure 2.17) is expressed as (Eq. (2.35) with Eqs. (2.33) and (2.52))

$$\mathbf{r},_{\eta} = \frac{1}{2}\xi(\Delta_x \mathbf{i} + \Delta_y \mathbf{j}) \tag{2.61}$$

On the boundary ($\xi = 1$), it is equal to

$$\mathbf{r}_{b,\eta} = \frac{1}{2}(\Delta_x \mathbf{i} + \Delta_y \mathbf{j}) \tag{2.62}$$

The infinitesimal length of S_c equals (Eqs. (2.38) and (2.62))

$$\mathrm{d}S_c = \frac{1}{2}\xi\sqrt{\Delta_x^2 + \Delta_y^2}\mathrm{d}\eta \tag{2.63}$$

It is easily verified that integrating over $-1 \le \eta \le +1$ leads to the length of a coordinate curve $\xi\sqrt{\Delta_x^2 + \Delta_y^2}$.
Substituting Eq. (2.62) into Eq. (2.40), the unit outward normal vector $\mathbf{n}_c$ of the coordinate curve S_c is expressed as

$$\mathbf{n}_c = \frac{\Delta_y}{\sqrt{\Delta_x^2 + \Delta_y^2}}\mathbf{i} - \frac{\Delta_x}{\sqrt{\Delta_x^2 + \Delta_y^2}}\mathbf{j} \tag{2.64}$$

The unit outward normal vector $\mathbf{n}_r$ of a coordinate curve S_r is

$$\mathbf{n}_r = -\frac{y_b}{r_b}\mathbf{i} + \frac{x_b}{r_b}\mathbf{j}$$

with x_b and y_b given in Eq. (2.49) and r_b in Eq. (2.6b). The equations of the unit outward normal vectors $\mathbf{n}_c$ and $\mathbf{n}_r$ can be verified by inspecting Figure 2.17.
The infinitesimal area dV is obtained by substituting Eq. (2.58) into Eq. (2.47)

$$\mathrm{d}V = \frac{1}{2}(x_1 y_2 - x_2 y_1)\xi \mathrm{d}\xi \mathrm{d}\eta \tag{2.65}$$

Integrating Eq. (2.65) over the domain defined by $0 \le \xi \le 1$ and $-1 \le \eta \le +1$ results in $(x_1 y_2 - x_2 y_1)/2$, which is the area of the triangular sector covered by scaling the 2-node element.

2.3 Governing Equations of Linear Elasticity in Scaled Boundary Coordinates

The spatial coordinates of governing equations for two-dimensional (2D) linear elasticity found in Section A.2 are transformed into the scaled boundary coordinates. The directions of displacement components $\{u\}$, and strain components $\{\varepsilon\}$ and stress components $\{\sigma\}$ are retained in the original Cartesian coordinate directions. This is analogous to the procedure for developing isoparametric elements in the standard finite element method (Cook et al., 2002; Zienkiewicz et al., 2005).

The spatial derivatives are transformed from the Cartesian coordinates x, y into the scaled boundary coordinates ξ, η in Eq. (2.23). Substituting this equation into the differential operator for 2D elasticity (Eq. (A.31)) and grouping the terms according to the partial derivatives results in

$$[L] = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} = [b_1]\frac{\partial}{\partial \xi} + \frac{1}{\xi}[b_2]\frac{\partial}{\partial \eta} \tag{2.66}$$

where the matrices

$$[b_1] = \frac{1}{|J_b|}\begin{bmatrix} y_{b,\eta} & 0 \\ 0 & -x_{b,\eta} \\ -x_{b,\eta} & y_{b,\eta} \end{bmatrix} \tag{2.67a}$$

$$[b_2] = \frac{1}{|J_b|}\begin{bmatrix} -y_b & 0 \\ 0 & x_b \\ x_b & -y_b \end{bmatrix} \tag{2.67b}$$

are introduced. Note that $[b_1]$ and $[b_2]$ depend only on the geometry of the element at the boundary and are independent of ξ. It is easy to verify from Eq. (2.67) that the following identity between $[b_1]$ and $[b_2]$ exists

$$(|J_b|[b_2])_{,\eta} = -|J_b|[b_1] \tag{2.68}$$

This will be used later in the derivation of the scaled boundary finite element equation.

Substituting Eq. (2.66) into Eq. (A.5), the strains are expressed in the scaled boundary coordinates as

$$\{\varepsilon\} = [b_1]\frac{\partial\{u\}}{\partial \xi} + \frac{1}{\xi}[b_2]\frac{\partial\{u\}}{\partial \eta} \tag{2.69}$$

The directions of the strain and stress components are in the Cartesian coordinates. The stress-strain relationship (Eq. (A.11)) and the elasticity matrix $[D]$ are not affected by the transformation of the spatial coordinates.

The surface tractions on the coordinate curves S_c and S_r are determined using Eq. (A.36) and the unit outward normal vectors in the scaled boundary coordinates. On a coordinate curve S_c (parallel to the boundary) with a constant radial coordinate ξ, the unit outward normal is given in Eq. (2.41). Replacing n_x and n_y in Eq. (A.36) with n_{cx} and n_{cy}, respectively, the surface traction $\{t_c\} = \{t_c(\eta)\}$ is expressed as

$$\{t_c\} = \frac{1}{|\mathbf{r}_{b,\eta}|}\begin{bmatrix} y_{b,\eta} & 0 & -x_{b,\eta} \\ 0 & -x_{b,\eta} & y_{b,\eta} \end{bmatrix}\{\sigma\} \tag{2.70}$$

Comparing Eq. (2.70) to Eq. (2.67a), it is identified that

$$\{t_c\} = \frac{|J_b|}{|\mathbf{r}_{b,\eta}|}[b_1]^T\{\sigma\} \tag{2.71}$$

applies.

Similarly, the surface tractions $\{t_r\} = \{t_r(\xi)\}$ on a coordinate curve S_r (a radial line passing through the scaling centre) are determined by substituting the components of the unit outward normal vector given in Eq. (2.45) into Eq. (A.36) and leads to

$$\{t_r\} = \frac{1}{r_b}\begin{bmatrix} -y_b & 0 & x_b \\ 0 & x_b & -y_b \end{bmatrix}\{\sigma\} \tag{2.72}$$

Replacing the matrix on the right-hand side of Eq. (2.72) using Eq. (2.67b) results in

$$\{t_r\} = \frac{|J_b|}{r_b}[b_2]^T\{\sigma\} \tag{2.73}$$

2.4 Semi-analytical Representation of Displacement and Strain Fields

The equations of equilibrium are expressed by substituting Eq. (2.66) into Eq. (A.10) as

$$[b_1]^T\frac{\partial\{\sigma\}}{\partial\xi} + \frac{1}{\xi}[b_2]^T\frac{\partial\{\sigma\}}{\partial\eta} = \{0\} \tag{2.74}$$

The displacement field in an S-element is represented semi-analytically in the scaled boundary finite element method. In the circumferential direction, the solution is given at discrete nodal points as in 1D finite elements. In the radial direction, the solution is obtained analytically.

The radial lines passing through the scaling centre O and a node at the boundary as shown in Figure 2.18a are addressed. The lines are numbered sequentially corresponding to the numbering of nodes at the boundary. Along a radial line i, the circumferential coordinate η is constant and nodal displacement functions $u_{ix}(\xi)$ and $u_{iy}(\xi)$ are introduced. They analytically describe the variation of the displacement components. Their directions are along the Cartesian coordinates x and y. All the nodal displacement functions of an S-element (Figure 2.18a) are assembled to construct a vector of nodal displacement functions

$$\{u(\xi)\} = [u_{1x}(\xi) \quad u_{1y}(\xi) \quad u_{2x}(\xi) \quad u_{2y}(\xi) \quad \dots \quad u_{ix}(\xi) \quad u_{iy}(\xi) \quad \dots]^T \tag{2.75}$$

On the boundary ($\xi = 1$), the nodal displacement functions are equal to the nodal displacements denoted as $\{d\}$

$$\{d\} = \{u(\xi = 1)\} \tag{2.76}$$

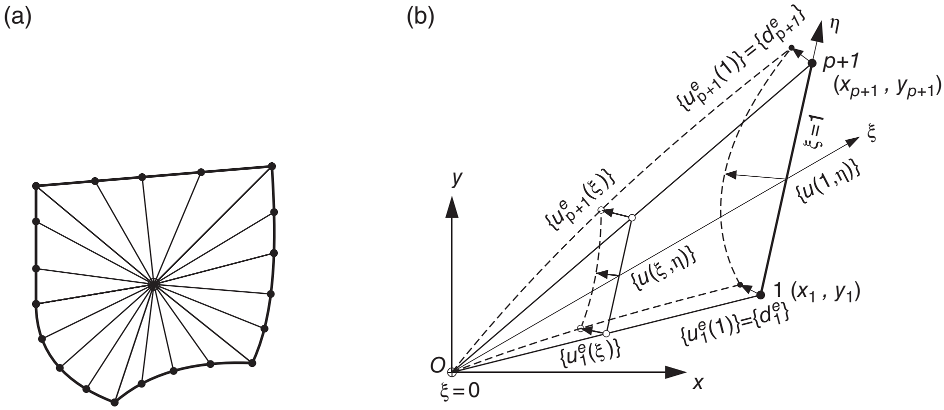


Figure 2.18 Semi-analytical representation of displacement field in an S-element by interpolating nodal displacement functions element-by-element independently. (a) Displacement functions are introduced along radial lines connecting the scaling centre and nodes at the boundary. (b) Displacement field in a sector covered by scaling one line element on boundary. The dashed lines show the deformed shapes. Displacements along the circumferential direction η are obtained by interpolating the nodal displacement functions.

The displacement field is obtained by interpolating the displacement functions in the circumferential direction η.

Similar to the formulation of standard displacement-based finite elements, the interpolation and other operations are performed element-by-element and independently of each other. A sector covered by the scaling of one isoparametric displacement-based p-th order element at the boundary is illustrated in Figure 2.18b, where only the two end nodes and one middle node are shown. The nodes are numbered sequentially from 1 to $p+1$ following the counter-clockwise direction. The element and a coordinate curve with constant radial coordinates ξ are plotted. The nodal displacement functions related to all the nodes of this element are assembled in a vector $\{u^e(\xi)\} = [u^e_{1x}(\xi) \quad u^e_{1y}(\xi) \quad \ldots \quad u^e_{(p+1)x}(\xi) \quad u^e_{(p+1)y}(\xi)]^T$, where the superscript e denotes functions of one element. The displacement functions $\{u^e_1(\xi)\} = [u^e_{1x}(\xi) \quad u^e_{1y}(\xi)]^T$ and $\{u^e_{p+1}(\xi)\} = [u^e_{(p+1)x}(\xi) \quad u^e_{(p+1)y}(\xi)]^T$ at the two end nodes of the element are depicted by dashed lines in Figure 2.18. The displacement functions, $\{u^e(\xi)\}$, of a line element are related to the displacement functions, $\{u(\xi)\}$, of the S-element via the element connectivity. The assembling of displacement functions of the line elements to form the displacement functions of the S-element is expressed as

$$\{u(\xi)\} = \sum_e \{u^e(\xi)\} \tag{2.77}$$

where the symbol $\sum_e$ indicates standard finite element assembly of all the line elements at the boundary.

The displacements $\{u\} = \{u(\xi,\eta)\} = [u_x(\xi,\eta) \quad u_y(\xi,\eta)]^T$ at a point (ξ,η) inside a sector are obtained by interpolating the displacement functions $\{u^e(\xi)\}$. At the specified value of the radial coordinate ξ, the displacement functions $u^e_{ix}(\xi)$ and $u^e_{iy}(\xi)$, where

$i = 1, 2, \ldots, p+1$, are evaluated. The displacements in the x- and y-directions at the specified circumferential coordinate η are interpolated independently

$$u_x = u_x(\xi, \eta) = \sum_{i=1}^{p+1} N_i u_{ix}^e(\xi) \tag{2.78a}$$

$$u_y = u_y(\xi, \eta) = \sum_{i=1}^{p+1} N_i u_{iy}^e(\xi) \tag{2.78b}$$

where $N_i = N_i(\eta)$ are the shape functions of the p-th order element. The deformed shapes of the element and coordinate curve are shown by dashed lines in Figure 2.18b.

Equation (2.78) is expressed in matrix form as

$$\{u\} = \{u(\xi, \eta)\} = [N_u]\{u^e(\xi)\} \tag{2.79}$$

with

$$[N_u] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \ldots & N_{p+1} & 0 \\ 0 & N_1 & 0 & N_2 & \ldots & 0 & N_{p+1} \end{bmatrix} \tag{2.80}$$

Substituting the displacement field in Eq. (2.79) into Eq. (2.69), the strain field $\{\varepsilon\} = \{\varepsilon(\xi, \eta)\}$ is expressed in the scaled boundary coordinates as

$$\{\varepsilon\} = [b_1][N_u]\{u^e(\xi)\},_\xi + \frac{1}{\xi}[b_2][N_u],_\eta \{u^e(\xi)\} \tag{2.81}$$

The strain-displacement matrices $[B_1] = [B_1(\eta)]$ and $[B_2] = [B_2(\eta)]$ are introduced

$$[B_1] = [b_1][N_u] \tag{2.82a}$$

$$[B_2] = [b_2][N_u],_\eta \tag{2.82b}$$

They depend on only the geometry of the line element. Using Eq. (2.82), the strain field in Eq. (2.81) is rewritten as

$$\{\varepsilon\} = [B_1]\{u^e(\xi)\},_\xi + \frac{1}{\xi}[B_2]\{u^e(\xi)\} \tag{2.83}$$

It is expressed semi-analytically using the displacement functions.

The stress field $\{\sigma\} = \{\sigma(\xi, \eta)\}$ follows from Eqs. (A.11) and (2.83) as

$$\{\sigma\} = [D]\left([B_1]\{u^e(\xi)\},_\xi + \frac{1}{\xi}[B_2]\{u^e(\xi)\}\right) \tag{2.84}$$

The surface tractions on a coordinate curve S_c with a constant radial coordinate ξ and a coordinate curve S_r with a constant radial coordinate η are given in Eqs. (2.71) and (2.73), respectively. Using Eq. (2.84), they can also be expressed in terms of the displacement functions.

2.5 Derivation of the Scaled Boundary Finite Element Equation by the Virtual Work Principle

The scaled boundary finite element equation is derived by converting the governing partial differential equations to a weak form in the circumferential direction only. This leads to a system of ordinary differential equations with the radial coordinate as the independent variable. For a linear elastostatic problem without the presence of any body force, the scaled boundary finite element equation is a system of homogeneous Euler-Cauchy ordinary differential equations and can be solved analytically by following standard procedures (Kreyszig, 2011). The scaled boundary finite element equation can be derived by applying either the Galerkin weighted residual method (Song and Wolf, 1997) or the virtual work principle (Deeks and Wolf, 2002d). Other techniques, such as the principle of stationary potential energy, should also be possible. In this section, the derivation based on the virtual work principle is presented.

2.5.1 Virtual Displacement and Strain Fields in Scaled Boundary Coordinates

In the principle of virtual work, also known as the principle of virtual displacements, a virtual displacement field $\{\delta u\} = \{\delta u(\xi, \eta)\} = [\delta u_x(\xi, \eta),\ \delta u_y(\xi, \eta)]^T$ is postulated. The visual displacement field satisfies the displacement boundary conditions and does not alter the stress field and external loads. It is represented analogously to the real displacement field in Section 2.4.

Virtual displacement functions $\{\delta u(\xi)\}$ are introduced on the radial lines connecting the scaling centre and the nodes at the boundary of the S-element. The value of the virtual displacement functions at the boundary ($\xi = 1$) is denoted as

$$\{\delta d\} = \{\delta u(\xi = 1)\} \tag{2.85}$$

The virtual displacement field $\{\delta u\}$ is obtained by interpolating the virtual displacement functions element-by-element as in Eq. (2.79) using the shape functions $[N_u]$ of a line element.

$$\{\delta u\} = [N_u]\{\delta u^e(\xi)\} \tag{2.86}$$

The virtual strain field $\{\delta\varepsilon\} = \{\delta\varepsilon(\xi, \eta)\} = [\delta\varepsilon_x(\xi, \eta),\ \delta\varepsilon_y(\xi, \eta),\ \delta\gamma_{xy}(\xi, \eta)]^T$ produced by the virtual displacement field is determined in the same way as the real strain field. For a sector covered by scaling a line element at the boundary (Figure 2.18). Using Eqs. (2.86) and (2.69), the virtual strain field is expressed as

$$\{\delta\varepsilon\} = [B_1]\{\delta u^e(\xi)\}_{,\xi} + \frac{1}{\xi}[B_2]\{\delta u^e(\xi)\} \tag{2.87}$$

where $[B_1]$ and $[B_2]$ are given in Eq. (2.82).

2.5.2 Nodal Force Functions

Corresponding to the nodal displacement functions, the nodal force functions are introduced on the radial lines connecting the scaling centre and the nodes on the boundary to represent the stress field.

A coordinate curve S_c defined by scaling a line element at the boundary with a constant ξ is shown in Figure 2.16. The normal vector $\mathbf{n}_c$ (Eq. (2.40)) is the outward normal vector of the sector covered by scaling the curve S_c to the scaling centre O. The surface tractions $\{t_c\}$ on curve S_c resulting from the stress field $\{\sigma\}$ are given in Eq. (2.71). For the sector covered by scaling the line element at the boundary, the nodal force functions $\{q^e(\xi)\}$ are defined as being equivalent in the sense of virtual work to the surface tractions $\{t_c\}$

$$\{\delta u^e(\xi)\}^T\{q^e(\xi)\} = \int_{S_c}\{\delta u\}^T\{t_c\}\mathrm{d}S_c \tag{2.88}$$

Substituting the virtual displacement field $\{\delta u\}$ (Eq. (2.86)) into Eq. (2.88) and considering the arbitrariness of the virtual displacement functions $\{\delta^e u(\xi)\}$, the nodal force functions are expressed as

$$\{q^e(\xi)\} = \int_{S_c}[N_u]^T\{t_c\}\mathrm{d}S_c \tag{2.89}$$

Substituting the surface traction (Eq. (2.71)) and the infinitesimal surface $\mathrm{d}S_c$ (Eq. (2.38)) into Eq. (2.89) yields

$$\begin{aligned}\{q^e(\xi)\} &= \int_{-1}^{+1}[N_u]^T\frac{|J_b|}{|\mathbf{r}_{b,\eta}|}[b_1]^T\{\sigma\}\xi|\mathbf{r}_{b,\eta}|\mathrm{d}\eta\\ &= \int_{-1}^{+1}[N_u]^T[b_1]^T\{\sigma\}\xi|J_b|\mathrm{d}\eta\end{aligned} \tag{2.90}$$

Using Eq. (2.82a), Eq. (2.90) is rewritten as

$$\{q^e(\xi)\} = \xi\int_{-1}^{+1}[B_1]^T\{\sigma\}|J_b|\mathrm{d}\eta \tag{2.91}$$

The nodal force vectors of individual line elements are assembled to form the nodal force vector of the S-element

$$\{q(\xi)\} = \sum_e\{q^e(\xi)\} \tag{2.92}$$

The nodal force functions $\{q(\xi)\}$ of the S-element at a specified radial coordinate ξ are similar to the nodal forces of an element in the standard finite element method. In particular, the nodal forces of the S-element (Figure 2.9) are given by $\{q(\xi = 1)\}$, i.e. the nodal force functions $\{q(\xi)\}$ at the boundary $\xi = 1$.

2.5.3 The Scaled Boundary Finite Element Equation

The virtual work principle is applied to a single S-element. A pentagon S-element is depicted in Figure 2.19 as an example. The boundary of the S-element is discretized with line elements. The nodal displacements and nodal forces are shown in Figure 2.19a and Figure 2.19b, respectively. In each sector defined by scaling a line element towards the scaling centre, the (virtual) displacement and strain fields are described in Section 2.4. For conciseness in the derivation of the scaled boundary finite element equation, only concentrated nodal forces are considered in this section (The surface tractions and body loads are addressed in Sections 3.4.2 and 3.9, respectively.)

The principle of virtual work is expressed for the S-element as

$$\underbrace{\int_V\{\delta\varepsilon\}^T\{\sigma\}\mathrm{d}V}_{U_\varepsilon} = \{\delta d\}^T\{F\} \tag{2.93}$$

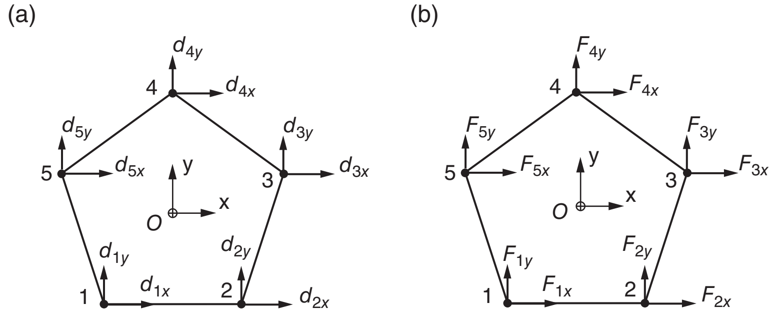


Figure 2.19 Nodal displacement and nodal forces of a pentagon S-element. (a) Nodal displacements. (b) Nodal forces.

where $\{\sigma\}$ is the stress field, and $\{F\}$ consists of the nodal forces (Figure 2.19b). The left-hand side is the virtual increment of internal strain energy, and the right-hand side the virtual increment of the work done by the external forces.

The virtual increment of the strain energy of the S-element at the left-hand side of Eq. (2.93) is evaluated sector-by-sector corresponding to the line elements at the boundary. Substituting the virtual strain field in Eq. (2.87) for a sector V^e into U_ε in Eq. (2.93) results in

$$U_\varepsilon = \underbrace{\sum \int_{V^e} \{\delta u^e(\xi)\}^T_{,\xi} [B_1]^T \{\sigma\} \mathrm{d}V}_{U_{\varepsilon I}} + \underbrace{\sum \int_{V^e} \{\delta u^e(\xi)\}^T [B_2]^T \{\sigma\} \frac{1}{\xi} \mathrm{d}V}_{U_{\varepsilon II}} \tag{2.94}$$

The two terms on the right-hand side of this equation are considered separately in the following.

The first term on the right-hand side of Eq. (2.94) is expressed, using Eq. (2.47) for dV in the scaled boundary coordinates, as

$$U_{\varepsilon I} = \sum \int_0^1 \{\delta u^e(\xi)\}^T_{,\xi} \int_{-1}^{+1} [B_1]^T \xi \{\sigma\} |J_b| \mathrm{d}\eta \mathrm{d}\xi \tag{2.95}$$

Replacing the summation with the finite element assembly, it is rewritten as

$$U_{\varepsilon I} = \int_0^1 \{\delta u(\xi)\}^T_{,\xi} \sum_e \xi \int_{-1}^{+1} [B_1]^T \{\sigma\} |J_b| \mathrm{d}\eta \mathrm{d}\xi \tag{2.96}$$

where the assembly of the virtual displacement vector $\{\delta u(\xi)\}$ of the S-element follows from Eq. (2.77). Considering Eq. (2.91), Eq. (2.96) is expressed as

$$U_{\varepsilon I} = \int_0^1 \{\delta u(\xi)\}^T_{,\xi} \sum_e \{q^e(\xi)\} \mathrm{d}\xi \tag{2.97}$$

Introducing the assembly of the nodal force functions in Eq. (2.92) leads to

$$U_{\varepsilon I} = \int_0^1 \{\delta u(\xi)\}^T_{,\xi} \{q(\xi)\} \mathrm{d}\xi \tag{2.98}$$

Applying integration by parts with respect to ξ to Eq. (2.98) results in

$$U_{\varepsilon I} = \left(\{\delta u(\xi)\}^T\{q(\xi)\}\right)_0^1 - \int_0^1 \{\delta u(\xi)\}^T\{q(\xi)\},_\xi \mathrm{d}\xi \tag{2.99}$$

At the lower limit $\xi = 0$, the first term on the left-hand side vanishes. Considering Eq. (2.85) at the upper limit of the first term, Eq. (2.99) is rewritten as

$$U_{\varepsilon I} = \{\delta d\}^T\{q(\xi = 1)\} - \int_0^1 \{\delta u(\xi)\}^T\{q(\xi)\},_\xi \mathrm{d}\xi \tag{2.100}$$

The second term on the right-hand side of Eq. (2.94) is also evaluated sector-by-sector

$$U_{\epsilon II} = \sum \int_0^1 \{\delta u^e(\xi)\}^T \int_{-1}^{+1} [B_2]^T\{\sigma\}|J_b|\mathrm{d}\eta\mathrm{d}\xi \tag{2.101}$$

Replacing the summation with finite element assembly, Eq. (2.101) is expressed as

$$U_{\epsilon II} = \int_0^1 \{\delta u(\xi)\}^T \sum_e \int_{-1}^{+1} [B_2]^T\{\sigma\}|J_b|\mathrm{d}\eta\mathrm{d}\xi \tag{2.102}$$

The virtual increment of the strain energy in Eq. (2.94) is obtained by substituting $U_{\varepsilon I}$ in Eq. (2.100) and $U_{\varepsilon II}$ in Eq. (2.102) as

$$U_\varepsilon = \{\delta d\}^T q(\xi = 1) - \int_0^1 \{\delta u(\xi)\}^T \left(\{q(\xi)\},_\xi - \sum_e \int_{-1}^{+1} [B_2]^T\{\sigma\}|J_b|\mathrm{d}\eta\right)\mathrm{d}\xi \tag{2.103}$$

Substituting Eq. (2.103) back into Eq. (2.93), the complete virtual work equation of the S-element is obtained

$$\begin{aligned}\{\delta d\}^T q(\xi = 1) - \int_0^1 \{\delta u(\xi)\}^T \left(\{q(\xi)\},_\xi - \sum_e \int_{-1}^{+1} [B_2]^T\{\sigma\}|J_b|\mathrm{d}\eta\right)\mathrm{d}\xi \\ = \{\delta d\}^T\{F\}\end{aligned} \tag{2.104}$$

This equation is to be satisfied for all admissible $\{\delta u(\xi)\}$ (on the boundary $\{\delta d\} = \{\delta u(\xi = 1)\}$ in Eq. (2.85) applies) in strong form along the radial direction ξ. This yields two conditions. At the boundary (boundary conditions),

$$\{F\} = \{q(\xi = 1)\} \tag{2.105}$$

has to hold. In the domain ($0 \le \xi < 1$), the integrand is set to zero leading to

$$\{q(\xi)\},_\xi = \sum_e \int_{-1}^{+1} [B_2]^T\{\sigma\}|J_b|\mathrm{d}\eta \tag{2.106}$$

where the nodal force functions are related to stresses $\{\sigma\}$ (Eq. (2.91)). This equation can be regarded as the equations of equilibrium expressed in a weak form in the circumferential direction η. When Eq. (2.106) is satisfied, the virtual increment of the strain energy, Eq. (2.103) is simply equal to (Eq. 2.105)

$$U_\varepsilon = \{\delta d\}^T\{q(\xi = 1)\} \tag{2.107}$$

i.e. the work done by the nodal forces at the boundary.

After replacing the stresses $\{\sigma\}$ using Eq. (2.84), Eqs. (2.91) and (2.106) are expanded and rearranged as

$$\{q(\xi)\} = \sum_e \xi \int_{-1}^{+1} [B_1]^T[D] \left([B_1]\{u^e(\xi)\},_\xi + \frac{1}{\xi}[B_2]\{u^e(\xi)\} \right) |J_b| \mathrm{d}\eta \tag{2.108}$$

and

$$\{q(\xi)\},_\xi = \sum_e \int_{-1}^{+1} [B_2]^T[D] \left([B_1]\{u^e(\xi)\},_\xi + \frac{1}{\xi}[B_2]\{u^e(\xi)\} \right) |J_b| \mathrm{d}\eta \tag{2.109}$$

respectively. The integrations with respect to η are separated from the nodal displacement functions, which depend on ξ only. The following coefficient matrices are introduced

$$[E_0] = \sum_e [E_0^e]; \qquad \text{with} \quad [E_0^e] = \int_{-1}^{+1} [B_1]^T[D][B_1]|J_b|\mathrm{d}\eta \tag{2.110a}$$

$$[E_1] = \sum_e [E_1^e]; \qquad \text{with} \quad [E_1^e] = \int_{-1}^{+1} [B_2]^T[D][B_1]|J_b|\mathrm{d}\eta \tag{2.110b}$$

$$[E_2] = \sum_e [E_2^e]; \qquad \text{with} \quad [E_2^e] = \int_{-1}^{+1} [B_2]^T[D][B_2]|J_b|\mathrm{d}\eta \tag{2.110c}$$

It is worthwhile noting that:

1) The coefficient matrices $[E_0]$, $[E_1]$ and $[E_2]$ of the S-element are obtained by assembling the element coefficient matrices $[E_0^e]$, $[E_1^e]$ and $[E_2^e]$ element-by-element according to the element connectivity.
2) The element coefficient matrices are evaluated on the individual line element at the boundary. They depend on only the geometry of the boundary, as represented by the line element.
3) The integration can be performed numerically in the same way as the computation of the stiffness matrix of one-dimensional finite elements. The rules for the choice of the order of integration quadrature in the conventional finite element method are equally applicable.
4) The relationship of the element coefficient matrices to the stiffness matrix of a two-dimensional element is shown in Section 5.2.3, Wolf and Song (1996).
5) Both $[E_0]$ and $[E_2]$ are symmetric. It is also shown later in Section 6.10.1 that $[E_0]$ is positive definite.

Equations (2.108) and (2.109) are simplified, by using the coefficient matrices in Eq. (2.110), as

$$\{q(\xi)\} = [E_0]\xi\{u(\xi)\},_\xi + [E_1]^T\{u(\xi)\} \tag{2.111a}$$

$$\xi\{q(\xi)\},_\xi = [E_1]\xi\{u(\xi)\},_\xi + [E_2]\{u(\xi)\} \tag{2.111b}$$

The nodal force functions $\{q(\xi)\}$ can be eliminated by substituting Eq. (2.111a) into Eq. (2.111b) leading to the scaled boundary finite element equation in displacement

$$[E_0]\xi^2\{u(\xi)\},_{\xi\xi} + ([E_0] + [E_1]^T - [E_1])\xi\{u(\xi)\}._\xi - [E_2]\{u(\xi)\} = 0 \tag{2.112}$$

It is a system of second-order ordinary differential equations with the dimensionless radial coordinate ξ as the independent variable. In the derivation, the governing partial differential equations of linear elasticity are weakened in the circumferential direction in the manner of the finite element method, while the strong form remains in the radial direction.

Example 2.2 Two-node Element: Element Coefficient Matrices

Evaluate the element coefficient matrices and the equivalent nodal forces of the 2-node element shown in Figure 2.17 in Example 2.1, where the scaled boundary transformation has been performed.

1) Substituting Eqs. (2.49) and (2.52) into Eq. (2.67) leads to

$$[b_1] = \frac{1}{|J_b|}[C_1] \tag{2.113a}$$

$$[b_2] = \frac{1}{|J_b|}[C_2] - \frac{\eta}{|J_b|}[C_1] \tag{2.113b}$$

with the abbreviations

$$[C_1] = \frac{1}{2}\begin{bmatrix} \Delta_y & 0 \\ 0 & -\Delta_x \\ -\Delta_x & \Delta_y \end{bmatrix} \tag{2.114a}$$

$$[C_2] = \begin{bmatrix} -\bar{y} & 0 \\ 0 & \bar{x} \\ \bar{x} & -\bar{y} \end{bmatrix} \tag{2.114b}$$

The matrices $[b_1]$ and $[b_2]$ can also be obtained by using their definitions in Eq. (2.66) and the transformation of derivatives from Cartesian coordinates to the scaled boundary coordinates given in Eq. (2.60) in Example 2.1.

2) Using Eq. (2.16), the shape function matrix (Eq. (2.80)) of the 2-node element is expressed as

$$\begin{aligned}[N_u] &= \frac{1}{2}\begin{bmatrix} 1-\eta & 0 & 1+\eta & 0 \\ 0 & 1-\eta & 0 & 1+\eta \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} + \frac{\eta}{2}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\end{aligned} \tag{2.115}$$

The derivative of the shape function matrix with respect to the natural coordinate η of the parent element is obtained as

$$[N_u]_{,\eta} = \frac{1}{2}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{2.116}$$

3) Substituting Eqs. (2.113), (2.115) and (2.116) into Eq. (2.82), the strain-displacement matrices are express as

$$\begin{aligned}[B_1] &= [b_1][N_u] \\ &= \frac{1}{2|J_b|}\left[[C_1] \; [C_1] \right] + \frac{\eta}{2|J_b|}\left[-[C_1] \; [C_1] \right]\end{aligned} \tag{2.117a}$$

$$\begin{aligned}[B_2] &= [b_2][N_u]_{,\eta} \\ &= \frac{1}{2|J_b|}\left[-[C_2] \; [C_2] \right] - \frac{\eta}{2|J_b|}\left[-[C_1] \; [C_1] \right]\end{aligned} \tag{2.117b}$$

4) The following abbreviations will be introduced below to simplify nomenclature

$$[Q_0] = \frac{1}{2|J_b|}[C_1]^T[D][C_1] \tag{2.118a}$$

$$[Q_1] = \frac{1}{2|J_b|}[C_2]^T[D][C_1] \tag{2.118b}$$

$$[Q_2] = \frac{1}{2|J_b|}[C_2]^T[D][C_2] \tag{2.118c}$$

Substituting Eq. (2.117a) into Eq. (2.110a) and using $\int_{-1}^{+1} 1 \times \mathrm{d}\eta = 0$, the element coefficient matrix $[E_0^e]$ is obtained

$$\begin{aligned}
[E_0^e] &= \int_{-1}^{+1} [B_1]^T[D][B_1]|J_b|\mathrm{d}\eta \\
&= \int_{-1}^{+1} \frac{1}{2|J_b|}\left[[C_1]\ [C_1]\right]^T [D]\frac{1}{2|J_b|}\left[[C_1]\ [C_1]\right]|J_b|\mathrm{d}\eta \\
&\quad + \int_{-1}^{+1} \frac{\eta}{2|J_b|}\left[-[C_1]\ [C_1]\right]^T [D]\frac{\eta}{2|J_b|}\left[-[C_1]\ [C_1]\right]|J_b|\mathrm{d}\eta \\
&= \frac{1}{2}\begin{bmatrix}[Q_0] & [Q_0]\\ [Q_0] & [Q_0]\end{bmatrix} \times 2 + \frac{1}{2}\begin{bmatrix}[Q_0] & -[Q_0]\\ -[Q_0] & [Q_0]\end{bmatrix} \times \frac{2}{3} \\
&= \frac{2}{3}\begin{bmatrix}2[Q_0] & [Q_0]\\ [Q_0] & 2[Q_0]\end{bmatrix}
\end{aligned} \tag{2.119a}$$

Similarly, the coefficient matrix $[E_1^e]$ is obtained by substituting Eq. (2.117) into Eq. (2.110b)

$$\begin{aligned}
[E_1^e] &= \int_{-1}^{+1} [B_2]^T[D][B_1]|J_b|\mathrm{d}\eta \\
&= \int_{-1}^{+1} \frac{1}{2|J_b|}\left[-[C_2]\ [C_2]\right]^T [D]\frac{1}{2|J_b|}\left[[C_1]\ [C_1]\right]|J_b|\mathrm{d}\eta \\
&\quad - \int \frac{\eta}{2|J_b|}\left[-[C_1]\ [C_1]\right]^T [D]\frac{\eta}{2|J_b|}\left[-[C_1]\ [C_1]\right]|J_b|\mathrm{d}\eta \\
&= \frac{1}{2}\begin{bmatrix}-[Q_1] & -[Q_1]\\ [Q_1] & [Q_1]\end{bmatrix} \times 2 - \frac{1}{2}\begin{bmatrix}[Q_0] & -[Q_0]\\ -[Q_0] & [Q_0]\end{bmatrix} \times \frac{2}{3} \\
&= \begin{bmatrix}-[Q_1] & -[Q_1]\\ [Q_1] & [Q_1]\end{bmatrix} - \frac{1}{3}\begin{bmatrix}[Q_0] & -[Q_0]\\ -[Q_0] & [Q_0]\end{bmatrix}
\end{aligned} \tag{2.119b}$$

and the coefficient matrix $[E_2^e]$ by substituting Eq. (2.117b) into Eq. (2.110c)

$$\begin{aligned}
[E_2^e] &= \int_{-1}^{+1} [B_2]^T[D][B_2]|J_b|\mathrm{d}\eta \\
&= \int_{-1}^{+1} \frac{1}{2|J_b|}\left[-[C_2]\ [C_2]\right]^T [D]\frac{1}{2|J_b|}\left[-[C_2]\ [C_2]\right]|J_b|\mathrm{d}\eta \\
&\quad + \int_{-1}^{+1} \frac{\eta}{2|J_b|}\left[-[C_1]\ [C_1]\right]^T [D]\frac{\eta}{2|J_b|}\left[-[C_1]\ [C_1]\right]|J_b|\mathrm{d}\eta
\end{aligned}$$

$$=\frac{1}{2}\begin{bmatrix} [Q_2] & -[Q_2] \\ -[Q_2] & [Q_2] \end{bmatrix} \times 2 + \frac{1}{2}\begin{bmatrix} [Q_0] & -[Q_0] \\ -[Q_0] & [Q_0] \end{bmatrix} \times \frac{2}{3}$$

$$= \begin{bmatrix} [Q_2] & -[Q_2] \\ -[Q_2] & [Q_2] \end{bmatrix} + \frac{1}{3}\begin{bmatrix} [Q_0] & -[Q_0] \\ -[Q_0] & [Q_0] \end{bmatrix} \tag{2.119c}$$

The MATLAB code, as a part of the accompanying computer program Platypus, to compute the coefficient matrix of a 2-node line element is presented in Section 2.6. It is used in the subsequent sections for the static analysis of 2D problems.

Example 2.3 A square S-element is shown in Figure 2.20. Each edge is modelled by 1 line element. The dimensions, nodal numbers and the line element numbers (in circle) are shown in Figure 2.20. The elasticity constants are: Young's modulus E and Poisson's ratio $\nu = 0$. Considering the plane stress states, determine the element coefficient matrices $[E_0^e]$, $[E_1^e]$ and $[E_2^e]$ for Element 2 using the equations derived for 2-node line element in Example 2.2.

1) The scaling centre is selected at the centre of the square S-element. The origin of the Cartesian coordinates is placed at the scaling centre. The nodal coordinates and connectivity of the line elements at the boundary of the S-element are listed in the tables below. Note that the line elements follow the counter-clockwise direction around the scaling centre.

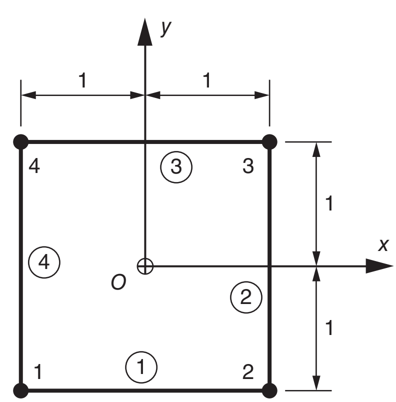
Nodal coordinates		
Node	***x*(m)**	***y*(m)**
1	−1	−1
2	1	−1
3	1	1
4	−1	1

Element connectivity		
Element	**Node 1**	**Node 2**
1	1	2
2	2	3
3	3	4
4	4	1

2) For Element 2, the coordinates of the two nodes are equal to

$$x_1 = 1; \quad y_1 = -1;$$
$$x_2 = 1; \quad y_2 = 1$$

Figure 2.20 A square S-element (Unit: meter).



Eqs. (2.50) and (2.51) lead to

$$\Delta_x = 0; \quad \Delta_y = 2;$$
$$\bar{x} = 1; \quad \bar{y} = 0$$

The determinant of the Jacobian matrix at the boundary (Eq. (2.58)) is equal to

$$|J_b| = \frac{1}{2}(1 \times 1 - 1 \times (-1)) = 1$$

The following two matrices are obtained from Eq. (2.114)

$$[C_1] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad [C_2] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

3) The elasticity matrix in plane stress state is expressed as (Eq. (A.42))

$$[D] = \frac{E}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4) Using $[C_1]$, $[C_2]$ and $[D]$ above, Equation (2.118) leads to

$$[Q_0] = \frac{1}{2 \times 1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\frac{E}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \frac{E}{4}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \frac{E}{4}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[Q_1] = \frac{1}{2 \times 1}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\frac{E}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \frac{E}{4}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \frac{E}{4}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[Q_2] = \frac{1}{2 \times 1}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\frac{E}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \frac{E}{4}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \frac{E}{4}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The coefficient matrices of Element 2 are obtained using Eq. (2.119) as

$$[E_0^e] = \frac{E}{6}\begin{bmatrix} 4 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned}[E_1^e] &= \frac{E}{4}\begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \frac{1}{3} \times \frac{E}{4}\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \\ -2 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \\ &= \frac{E}{12}\begin{bmatrix} -2 & -3 & 2 & -3 \\ 0 & -1 & 0 & 1 \\ 2 & 3 & -2 & 3 \\ 0 & 1 & 0 & -1 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}[E_2^e] &= \frac{E}{4}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 \\ -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} + \frac{1}{3} \times \frac{E}{4}\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \\ -2 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \\ &= \frac{E}{12}\begin{bmatrix} 5 & 0 & -5 & 0 \\ 0 & 7 & 0 & -7 \\ -5 & 0 & 5 & 0 \\ 0 & -7 & 0 & 7 \end{bmatrix}\end{aligned}$$

2.6 Computer Program Platypus: Coefficient Matrices of an S-element

MATLAB functions for computing the coefficient matrices of an S-element in the computer program Platypus accompanying this book are listed in this section. The use of the functions is demonstrated by examples. The boundary of the S-element is divided into 2-node line elements. The equations for computing the element coefficient matrices are obtained by performing the integrations analytically in Example 2.1. The origin of the coordinates has to be placed at the scaling centre in these functions. This requirement can be easily met by a translation of the coordinate system as shown later in Code List 3.2.

2.6.1 Element Coefficient Matrices of a 2-node Line Element

The function `EleCoeff2NodeEle.m` computes the element coefficient matrices $[E_0^e]$, $[E_1^e]$ and $[E_2^e]$ of a 2-node line element. It is listed below in Code List 2.1. The function is documented with comments in the list. The equation numbers in the comments refer to the equations in the text of this book. Additional explanations and an example to use this function are provided after the code list.

Code List 2.1: Element coefficient matrices

```
1  function [ e0, e1, e2, m0 ] = EleCoeff2NodeEle( xy, mat)
2  %Coefficient matrices of a 2-node line element
3  %
4  %  Inputs:
5  %   xy(i,:)    - coordinates of node i (orgin at scaling centre).
6  %                The nodes are numbered locally within
7  %                each line element
8  %   mat        - material constants
9  %     mat.D    - elasticity matrix
10 %     mat.den  - mass density
11 %
12 %  Outputs:
13 %    e0, e1, e2, m0  - element coefficient matrices
14
15 dxy = xy(2,:)-xy(1,:);  % .......................... [Δx, Δy], Eq. (2.50)
16 mxy = sum(xy)/2;  % .................................... [x̄, ȳ], Eq. (2.51)
17 a = xy(1,1)*xy(2,2)-xy(2,1)*xy(1,2);  % ............ a = 2|Jb|, Eq. (2.58)
18 if a < 1.d-10
19     disp('negative area (EleCoeff2NodeEle)');
20     pause
21 end
22 C1 = 0.5*[ dxy(2) 0; 0 -dxy(1); -dxy(1) dxy(2)];  % ....... Eq. (2.114a)
23 C2 = [-mxy(2) 0; 0 mxy(1); mxy(1) -mxy(2)];  % ........... Eq. (2.114b)
24 Q0 = 1/a*(C1'*mat.D*C1);  % ............................... Eq. (2.118a)
25 Q1 = 1/a*(C2'*mat.D*C1);  % ............................... Eq. (2.118b)
26 Q2 = 1/a*(C2'*mat.D*C2);  % ............................... Eq. (2.118c)
27 e0 =  2/3*[2*Q0 Q0; Q0 2*Q0];  % .......................... Eq. (2.119a)
28 e1 = -1/3*[ Q0 -Q0; -Q0  Q0] + [-Q1 -Q1;  Q1 Q1];  % ...... Eq. (2.119b)
29 e2 =  1/3*[ Q0 -Q0; -Q0  Q0] + [ Q2 -Q2; -Q2 Q2];  % ...... Eq. (2.119c)
30
31 % ...  mass coefficent matrix, Eq. (3.112)
32 m0 = a*mat.den/6*[ 2 0  1 0; 0 2  0 1; 1 0  2 0; 0 1 0 2 ];
33
34 end
```

The inputs are the nodal coordinates (argument `xy`) of the two nodes of the line element and the material properties (argument `mat`) given by the elasticity matrix (field `D`) and mass density (field `den`). The two nodes are numbered locally within the element (Figure 2.10 on page 38) and the direction from node 1 to node 2 has to follow the counter-clockwise direction around the scaling centre (Figure 2.9). The x and y coordinates of a node are stored as one row, i.e. the first row of `xy` is $[x_1, y_1]$ and the second one $[x_2, y_2]$.

The coefficient $[M_0^e]$, which is related to the mass matrix, is also calculated (see the code starting from Line 31). It is given in Eq. (3.98) for the dynamic analysis in Section 3.10.

Example 2.4 Use the function `EleCoeff2NodeEle` in Code List 2.1 on page 64 to compute the element coefficient matrices $[E_0]$, $[E_1]$ and $[E_2]$ of the four 2-node line elements in the square S-element in Example 2.3 on page 61. Assume the Young's modulus is equal to $E = 10\,\text{GPa}$.

The following MATLAB script calls the function `EleCoeff2NodeEle` to calculate the coefficient matrices of Element 1

```
%% Compute element coefficient matrices
% nodal coordinates of element given as [x1 y1; x2 y2]
xy = [-1 -1; 1 -1] % Element 1
% xy = [ 1 -1; 1 1] % Element 2
% xy = [ 1 1; -1 1] % Element 3
% xy = [-1 1; -1 -1] % Element 4

% elascity matrix (plane stress).
% E: Young's modulus; p: Poisson's ratio
ElasMtrx = @(E, p) E/(1-p^2)*[1 p 0; p 1 0; 0 0 (1-p)/2];
mat.D = ElasMtrx(10, 0); % E in GPa
mat.den = 2; % mass density in Mg per cubic meter

[ e0, e1, e2 ] = EleCoeff2NodeEle( xy, mat)
```

The above MATLAB script can be obtained by scanning the QR-code to the right.

The coordinates of the 4 nodes and the connectivity of the 4 line elements of the square S-element are given in Example 2.3. Nodes 1 and 2 of line element 1 are Nodes 1 $(-1, -1)$ and 2 $(1, -1)$ of the square S-element, respectively.

The elasticity matrix of the plane stress conditions is calculated by a so-called anonymous function `ElasMtrx`. It takes the Young's modulus `E` and Poisson's ratio `p` as inputs. The mass density `den` is set to zero as the mass matrix is not used in this example.

The MATLAB output is listed below

```
xy =
    -1    -1
     1    -1
```

```
e0 =
    3.3333         0    1.6667         0
         0    6.6667         0    3.3333
    1.6667         0    3.3333         0
         0    3.3333         0    6.6667

e1 =
   -0.8333         0    0.8333         0
    2.5000   -1.6667    2.5000    1.6667
    0.8333         0   -0.8333         0
   -2.5000    1.6667   -2.5000   -1.6667

e2 =
    5.8333         0   -5.8333         0
         0    4.1667         0   -4.1667
   -5.8333         0    5.8333         0
         0   -4.1667         0    4.1667
```

Element 2 is addressed. Nodes 1 and 2 of Element 2 are Nodes 2 $(1, -1)$ and 3 $(1, 1)$ of the square S-element, respectively. To obtain the coefficient matrices of Element 2, it is sufficient to replace the nodal coordinates of Element 1 in the previous MATLAB script with those of Element 2:

```
xy = [ 1 -1; 1 1] % Element 2
```

The MATLAB outputs are as follows:

```
xy =
     1    -1
     1     1

e0 =
    6.6667         0    3.3333         0
         0    3.3333         0    1.6667
    3.3333         0    6.6667         0
         0    1.6667         0    3.3333

e1 =
   -1.6667   -2.5000    1.6667   -2.5000
         0   -0.8333         0    0.8333
    1.6667    2.5000   -1.6667    2.5000
         0    0.8333         0   -0.8333

e2 =
    4.1667         0   -4.1667         0
         0    5.8333         0   -5.8333
   -4.1667         0    4.1667         0
         0   -5.8333         0    5.8333
```

Note that the same results of the coefficient matrices in Example 2.3 are obtained. Similarly, the coefficient matrices of Elements 3 and 4 are obtained by inputting their nodal coordinates

```
xy = [ 1 1; -1 1] % Element 3
```

for Element 3, and

```
xy = [-1 1; -1 -1] % Element 4
```

for Element 4. For the sake of brevity, only the result of the matrix $[E_0^e]$, which is used in the next section to illustrate the assembling process, is given below. The MATLAB output for Element 3 is

```
e0 =
    3.3333         0    1.6667         0
         0    6.6667         0    3.3333
    1.6667         0    3.3333         0
         0    3.3333         0    6.6667
```

and for Element 4 is

```
e0 =
    6.6667         0    3.3333         0
         0    3.3333         0    1.6667
    3.3333         0    6.6667         0
         0    1.6667         0    3.3333
```

2.6.2 Assembly of Coefficient Matrices of an S-element

The coefficient matrices of an S-element are obtained by assembling those of the line elements. The process is identical to the assembly of the element stiffness matrices of one-dimensional elements in the standard finite element method. The MATLAB function for assembly is listed below in Code List 2.2, followed by further explanations and an example.

Code List 2.2: Assembly of Coefficient Matrices of an S-element

```
function [ E0, E1, E2, M0 ] = SElementCoeffMtx(xy, conn, mat)
%Coefficient matrix of an S-element
%
% Inputs:
%   xy(i,:)     - coordinates of node i (orgin at scaling centre)
%                 The nodes are numbered locally within
%                 an S-element starting from 1
%   conn(ie,:) - local connectivity matrix of line element ie
%                 in the local nodal numbers of an S-element
%   mat         - material constants
%     mat.D     - elasticity matrix
%     mat.den   - mass density
%
% Outputs:
%   E0, E1, E2  - coefficient matrices of S-element

nd = 2*size(xy,1); % number of DOFs at boundary (2 DOFs per node)
% ... initializing variables
E0 = zeros(nd, nd);
E1 = zeros(nd, nd);
E2 = zeros(nd, nd);
M0 = zeros(nd, nd);
for ie = 1:size(conn,1)  % ............. loop over elements at boundary
    xyEle = xy(conn(ie,:),:);  % ...... nodal coordinates of an element
    % ... get element coefficient matrices of an element
    [ ee0, ee1, ee2, em0 ] = EleCoeff2NodeEle(xyEle, mat);
    % ... local DOFs (in S-element) of an element
    d = reshape([2*conn(ie,:)-1; 2*conn(ie,:)], 1, []);
    % ... assemble coefficient matrices of S-element
    E0(d,d) = E0(d,d) + ee0;
```

```
31      E1(d,d) = E1(d,d) + ee1;
32      E2(d,d) = E2(d,d) + ee2;
33      M0(d,d) = M0(d,d) + em0;
34  end
35
36  end
```

The inputs of this function include the nodal coordinates (argument `xy`), the connectivity table (argument `conn`) of the line elements and the same material constants as in Code List 2.1 on page 64. The coordinates (x_i, y_i) of node i are stored as the i-th row of the argument `xy`. The two nodes of a line element are stored as one row of the argument `conn`. This function loops over all the line elements to compute element coefficient matrices (by calling the function in Code List 2.1 on page 64) and assembles them to form the coefficient matrices of the S-element. At each node, two degrees of freedom (DOF) are considered. The vector `d` contains the connectivity between the DOFs of a 2-node line element and the DOFs of the S-element. It is constructed by the code on Line 28, which is explained in the following example. The assembling is performed according to the connectivity of DOFs.

Example 2.5 Use the function in Code List 2.2 to compute the coefficient matrices $[E_0]$, $[E_1]$ and $[E_2]$ of the square S-element in Example 2.4 on page 65. Assume the Young's modulus is equal to $E = 10\,\text{GPa}$ and Poisson's ratio $\nu = 0$.

The following MATLAB script calls the function `SElementCoeffMtx` to calculate the coefficient matrices of the S-element.

```
%% Compute coefficient matrices of square S-element
xy = [-1 -1; 1 -1; 1 1; -1 1]
conn = [1:4; 2:4 1]'
% elascity matrix (plane stress).
% E: Young's modulus; p: Poisson's ratio
ElasMtrx = @(E, p) E/(1-p^2)*[1 p 0; p 1 0; 0 0 (1-p)/2];
mat.D = ElasMtrx(10, 0); % E in GPa
mat.den = 2; % mass density in Mg per cubic meter
[ E0, E1, E2 ] = SElementCoeffMtx(xy, conn, mat)
```

The above MATLAB script can be obtained by scanning the QR-code to the right.

The nodal coordinates are displayed as

```
xy =
    -1    -1
     1    -1
     1     1
    -1     1
```

and the connectivity table as

```
conn =
     1     2
     2     3
     3     4
     4     1
```

The assembling process is detailed for Element 4, which corresponds to `ie=4` in the for-loop. The two nodes of Element 4 is given by

```
conn(ie,:)
ans =
     4     1
```

The nodal coordinates of an element are extracted from the nodal coordinates of the S-element (Line 24) as

```
xyEle =
    -1     1
    -1    -1
```

and used to compute the element coefficient matrices (see Example 2.4).

Within an S-element, the DOFs are numbered locally starting from the first node of the S-element. The two DOFs of the i-th node of the S-element are thus $(2i - 1)$ for u_x and $2i$ for u_y. The connectivity of the DOFs of a line element is constructed in Line 28. The DOFs for u_x and u_y of the nodes of the element are formed, respectively, as the first and second rows of a matrix

```
[2*conn(ie,:)-1; 2*conn(ie,:)]
ans =
     7     1
     8     2
```

It is reshaped column-wise into one row as

```
d =
     7     8     1     2
```

Line 30 assembles the entries of an element coefficient matrix $[E_0^e]$ to the entries of the same DOFs in the coefficient matrix $[E_0]$ of the S-element according to the connectivity of the DOFs. For example, `E0(d,d)` is expanded as:

$$\text{Element 4:}\quad \texttt{E0(d,d)} \Rightarrow \begin{bmatrix} \texttt{E0(7,7)} & \texttt{E0(7,8)} & \texttt{E0(7,1)} & \texttt{E0(7,2)} \\ \texttt{E0(8,7)} & \texttt{E0(8,8)} & \texttt{E0(8,1)} & \texttt{E0(8,2)} \\ \texttt{E0(1,7)} & \texttt{E0(1,8)} & \texttt{E0(1,1)} & \texttt{E0(1,2)} \\ \texttt{E0(2,7)} & \texttt{E0(2,8)} & \texttt{E0(2,1)} & \texttt{E0(2,2)} \end{bmatrix}$$

to which the coefficient matrix $[E_0^e]$ of Element 4 is added to.

Following the same steps, the connectivity of the DOFs of the other three elements are obtained as:

Element 1: `d = [ 1   2   3   4]`

Element 2: `d = [ 3   4   5   6]`

Element 3: `d = [ 5   6   7   8]`

Their element coefficient matrices $[E_0^e]$ are assembled, respectively, to the following entries of the coefficient matrix of the S-element:

$$\text{Element 1:}\quad \begin{bmatrix} \texttt{E0(1,1)} & \texttt{E0(1,2)} & \texttt{E0(1,3)} & \texttt{E0(1,4)} \\ \texttt{E0(2,1)} & \texttt{E0(2,2)} & \texttt{E0(2,3)} & \texttt{E0(2,4)} \\ \texttt{E0(3,1)} & \texttt{E0(3,2)} & \texttt{E0(3,3)} & \texttt{E0(3,4)} \\ \texttt{E0(4,1)} & \texttt{E0(4,2)} & \texttt{E0(4,3)} & \texttt{E0(4,4)} \end{bmatrix}$$

$$\text{Element 2:}\quad \begin{bmatrix} \texttt{E0(3,3)} & \texttt{E0(3,4)} & \texttt{E0(3,5)} & \texttt{E0(3,6)} \\ \texttt{E0(4,3)} & \texttt{E0(4,4)} & \texttt{E0(4,5)} & \texttt{E0(4,6)} \\ \texttt{E0(5,3)} & \texttt{E0(5,4)} & \texttt{E0(5,5)} & \texttt{E0(5,6)} \\ \texttt{E0(6,3)} & \texttt{E0(6,4)} & \texttt{E0(6,5)} & \texttt{E0(6,6)} \end{bmatrix}$$

$$\text{Element 3:}\quad \begin{bmatrix} \texttt{E0(5,5)} & \texttt{E0(5,6)} & \texttt{E0(5,7)} & \texttt{E0(5,8)} \\ \texttt{E0(6,5)} & \texttt{E0(6,6)} & \texttt{E0(6,7)} & \texttt{E0(6,8)} \\ \texttt{E0(7,5)} & \texttt{E0(7,6)} & \texttt{E0(7,7)} & \texttt{E0(7,8)} \\ \texttt{E0(8,5)} & \texttt{E0(8,6)} & \texttt{E0(8,7)} & \texttt{E0(8,8)} \end{bmatrix}$$

Using the element coefficient matrices $[E_0^e]$ obtained in Example 2.4 on page 65 for the four elements, the coefficient matrix of the S-element is obtained as

$$[E_0] = \begin{bmatrix} 10.00 & 0.00 & 1.67 & 0.00 & 0.00 & 0.00 & 3.33 & 0.00 \\ 0.00 & 10.00 & 0.00 & 3.33 & 0.00 & 0.00 & 0.00 & 1.67 \\ 1.67 & 0.00 & 10.00 & 0.00 & 3.33 & 0.00 & 0.00 & 0.00 \\ 0.00 & 3.33 & 0.00 & 10.00 & 0.00 & 1.67 & 0.00 & 0.00 \\ 0.00 & 0.00 & 3.33 & 0.00 & 10.00 & 0.00 & 1.67 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.67 & 0.00 & 10.00 & 0.00 & 3.33 \\ 3.33 & 0.00 & 0.00 & 0.00 & 1.67 & 0.00 & 10.00 & 0.00 \\ 0.00 & 1.67 & 0.00 & 0.00 & 0.00 & 3.33 & 0.00 & 10.00 \end{bmatrix}$$

In the same way, the coefficient matrices $[E_1]$ and $[E_2]$ of the S-element can be determined by performing the assembly. For the sake of brevity, the intermediate results are omitted. Only the final results are provided below:

$$[E_1] = \begin{bmatrix} -2.50 & 2.50 & 0.83 & 0.00 & 0.00 & 0.00 & 1.67 & 2.50 \\ 2.50 & -2.50 & 2.50 & 1.67 & 0.00 & 0.00 & 0.00 & 0.83 \\ 0.83 & 0.00 & -2.50 & -2.50 & 1.67 & -2.50 & 0.00 & 0.00 \\ -2.50 & 1.67 & -2.50 & -2.50 & 0.00 & 0.83 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.67 & 2.50 & -2.50 & 2.50 & 0.83 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.83 & 2.50 & -2.50 & 2.50 & 1.67 \\ 1.67 & -2.50 & 0.00 & 0.00 & 0.83 & 0.00 & -2.50 & -2.50 \\ 0.00 & 0.83 & 0.00 & 0.00 & -2.50 & 1.67 & -2.50 & -2.50 \end{bmatrix}$$

$$[E_2] = \begin{bmatrix} 10.00 & 0.00 & -5.83 & 0.00 & 0.00 & 0.00 & -4.17 & 0.00 \\ 0.00 & 10.00 & 0.00 & -4.17 & 0.00 & 0.00 & 0.00 & -5.83 \\ -5.83 & 0.00 & 10.00 & 0.00 & -4.17 & 0.00 & 0.00 & 0.00 \\ 0.00 & -4.17 & 0.00 & 10.00 & 0.00 & -5.83 & 0.00 & 0.00 \\ 0.00 & 0.00 & -4.17 & 0.00 & 10.00 & 0.00 & -5.83 & 0.00 \\ 0.00 & 0.00 & 0.00 & -5.83 & 0.00 & 10.00 & 0.00 & -4.17 \\ -4.17 & 0.00 & 0.00 & 0.00 & -5.83 & 0.00 & 10.00 & 0.00 \\ 0.00 & -5.83 & 0.00 & 0.00 & 0.00 & -4.17 & 0.00 & 10.00 \end{bmatrix}$$