GRAPHICAL

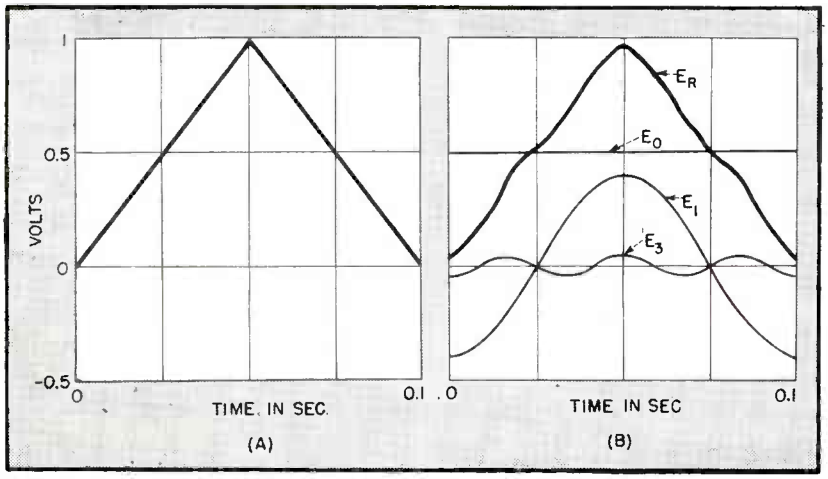

FIG. 1—Example showing how addition of first three principal components E_0, E_1, and E_3 gives a reasonably close approximation E_R to a pure triangular wave

Simple procedure for determining equation of wave and harmonic amplitudes up to the sixth by measuring ordinates on a cathode-ray oscilloscope screen or elsewhere, then substituting measured values in 6, 8, or 12-point schedules requiring only use of arithmetic

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THE determination of the d-c and various a-c components of a complex or distorted wave (such as might be seen on an oscilloscope screen) is frequently of considerable importance and value to both the student and the engineer. The graphical method presented here will yield this information with sufficient detail and accuracy for all ordinary needs, is purely arithmetical, and requires no long or involved calculations.

Principles of Method

Any sine-wave voltage making a specified angle with the time base can be represented by a sine-wave voltage (in phase with the time base) and a cosine-wave voltage (a sine wave 90° out of phase with the time base), each having the proper magnitude. The reason for the utilization of this principle is that often one of the sine-wave components of a complex wave is out of phase with the time base of the complex wave. In the analysis to follow, such a component will appear as a sine and a cosine term, and to obtain the magnitude of the actual component (and its phase angle if desired), these two terms are simply added vectorially.

Any continuous and periodic complex wave of voltage (or current) can be duplicated by adding together a number of separate voltages (or currents), each of which is a simple sine or cosine wave form of proper magnitude and frequency plus, in some cases, a d-c voltage (or current). Further, the frequencies of these a-c voltages will be integral multiples (harmonics) of the frequency of the complex wave.

As an example, consider the triangular wave shown in Fig. 1A having a peak value of 1 volt and a period of 1/10 sec (frequency 10 cycles). The Fourier analysis of this particular wave shows its principal components to be a d-c voltage of 0.5 volt, an a-c voltage of 0.405 volt at 10 cycles with a phase angle of $-90°$, and an a-c voltage of 0.045 volt at 30 cycles with a phase angle of $-90°$.

Figure 1B shows the results of adding together these first three principal components, which gives a reasonably good approximation of the original triangular wave. Of course the wave contains an infinite number of higher-order components other than those named, but they are of rapidly decreasing magnitude. (The next two, the 5th and 7th, have magnitudes of 0.0162 and 0.00827 respectively).

We may summarize these principles by stating that the equation of the general complex wave may be written in the form

$$E_R = E_0 + (e_1 \sin 2\pi ft + E_1 \cos 2\pi ft) + (e_2 \sin 2\pi 2ft + E_2 \cos 2\pi 2ft) + (e_3 \sin 2\pi 3ft) + E_3 \cos 2\pi 3ft) + \ldots + (e_n \sin 2\pi nft) + E_n \cos 2\pi nft)$$

ANALYSIS OF COMPLEX WAVES

where E_0 is the d-c component, $e_1 + jE_1$ is the fundamental a-c component (having the same period as that of the complex wave), $e_2 + jE_2$ is the second harmonic a-c component (having a period one-half that of the complex wave), etc. In a large number of practical cases, we are only interested in the first two or three a-c components of a complex wave because the magnitudes of the higher-order components are small.

Utilizing the above formula and the principles just discussed, let us analyze one cycle of the general complex wave of period $1/f$ sec (frequency f), shown in Fig. 2, for the d-c and first three a-c components.

Analysis of General Complex Wave

First, we divide the period of one cycle of the complex wave ($1/f$ sec) into six equal intervals of $1/6f$ sec each. From the graph, we measure the instantaneous values of the complex wave at these time intervals, noting the respective values as Y_1, Y_2, Y_3, etc., and tabulate as follows:

Time (sec)	$\frac{1}{6f}$	$\frac{2}{6f}$	$\frac{3}{6f}$	$\frac{4}{6f}$	$\frac{5}{6f}$	$\frac{1}{f}$
Inst. values	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6

Since the value of the complex wave at any instant must be equal to the algebraic sum of its components at that instant, we may write the following equations

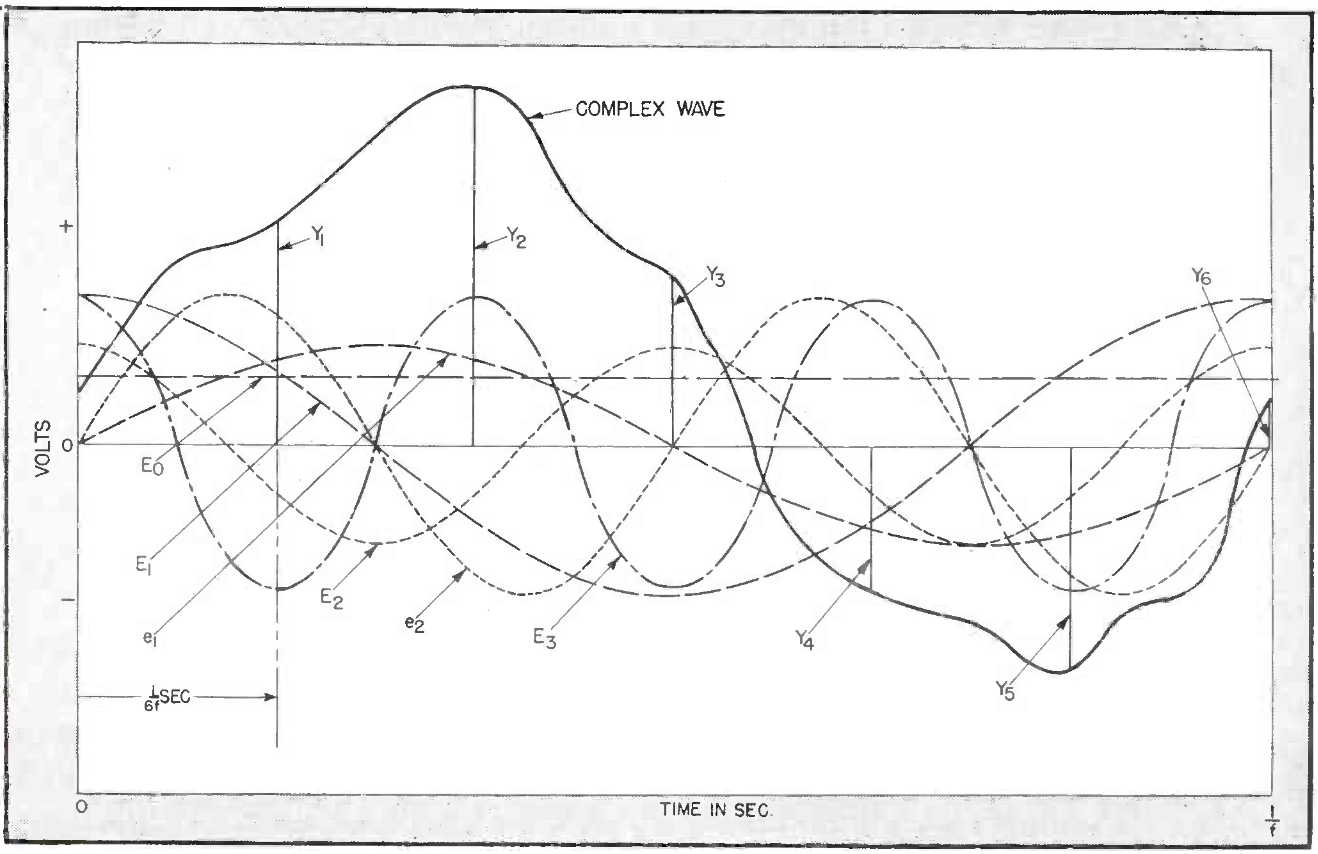

FIG. 2—Complex wave, showing method of measuring ordinates for a six-point analysis. The d-c component and the five sine-wave components are drawn in here only for illustrative purposes without any attempt to show correct amplitudes, and are not needed in the graphical analysis described

At $t = 1/6f$ sec:

$$E_R = Y_1 = E_0 + e_1 \sin 2\pi f \times 1/6f + E_1 \cos 2\pi f \times 1/6f + e_2 \sin 2\pi \times 2f \times 1/6f + E_2 \cos 2\pi \times 2f \times 1/6f + E_3 \cos 2\pi \times 3f \times 1/6f$$

At $t = 2/6f$ sec:

$$E_R = Y_2 = E_0 + e_1 \sin 2\pi f \times 2/6f + E_1 \cos 2\pi f \times 2/6f + e_2 \sin 2\pi \times 2f \times 2/6f + E_2 \cos 2\pi \times 2f \times 2/6f + E_3 \cos 2\pi \times 3f \times 2/6f$$

At $t = 3/6f$ sec:

$$E_R = Y_3 = E_0 + e_1 \sin 2\pi f \times 3/6f + E_1 \cos 2\pi f \times 3/6f + e_2 \sin 2\pi \times 2f \times 3/6f + E_2 \cos 2\pi \times 2f \times 3/6f + E_3 \cos 2\pi \times 3f \times 3/6f$$

At $t = 4/6f$ sec:

$$E_R = Y_4 = E_0 + e_1 \sin 2\pi f \times 4/6f + E_1 \cos 2\pi f \times 4/6f + e_2 \sin 2\pi \times 2f \times 4/6f + E_2 \cos 2\pi \times 2f \times 4/6f + E_3 \cos 2\pi \times 3f \times 4/6f$$

At $t = 5/6f$ sec:

$$E_R = Y_5 = E_0 + e_1 \sin 2\pi f \times 5/6f + E_1 \cos 2\pi f \times 5/6f + e_2 \sin 2\pi \times 2f \times 5/6f + E_2 \cos 2\pi \times 2f \times 5/6f + E_3 \cos 2\pi \times 3f \times 5/6f$$

At $t = 1/f$ sec:

$$E_R = Y_6 = E_0 + e_1 \sin 2\pi f \times 1/f + E_1 \cos 2\pi f \times 1/f + e_2 \sin 2\pi \times 2f \times 1/f + E_2 \cos 2\pi \times 2f \times 1/f + E_3 \cos 2\pi \times 3f \times 1/f$$

These six equations become, after simplification and substitution of trigonometric values

$$Y_1 = E_0 + (1/2)E_1 + (\sqrt{3}/2)e_1 - (1/2)E_2 + (\sqrt{3}/2)e_2 - E_3$$

$$Y_2 = E_0 - (1/2)E_1 + (\sqrt{3}/2)e_1 - (1/2)E_2 - (\sqrt{3}/2)e_2 + E_3$$

$$Y_3 = E_0 - E_1 + E_2 - E_3$$

$$Y_4 = E_0 - (1/2)E_1 - (\sqrt{3}/2)e_1 - (1/2)E_2 + (\sqrt{3}/2)e_2 + E_3$$

$$Y_5 = E_0 + (1/2)E_1 - (\sqrt{3}/2)e_1 - (1/2)E_2 - (\sqrt{3}/2)e_2 - E_3$$

$$Y_6 = E_0 + E_1 + E_2 + E_3$$

We now have six linear simultaneous equations containing six unknowns E_0, E_1, e_1, E_2, e_2, and E_3. (The Y values are known, having been measured from the graph.) When solved, these equations give the values set forth in Table I for a six-point analysis. Corresponding formulas for the eight and twelve-point methods, also given in Table I, are derived in the same manner and are used when it is necessary to take a greater number of points in order to secure more detailed and accurate results. The d-c, fundamental, and harmonic components are obtained from these values in the manner indicated at the bottom of the table.

TABLE I. SCHEDULES FOR GRAPHICAL ANALYSIS OF COMPLEX WAVES

SIX-POINT SCHEDULE—for d-c, 1st, 2nd, and 3rd harmonics

$$E_0 = (1/6)\,(Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6)$$
$$E_1 = (1/6)\,(Y_1 - Y_2 - Y_4 + Y_5) + (1/3)\,(-Y_3 + Y_6)$$
$$e_1 = (\sqrt{3}/6)\,(Y_1 + Y_2 - Y_4 - Y_5)$$
$$E_2 = (1/6)\,(-Y_1 - Y_2 - Y_4 - Y_5) + (1/3)\,(Y_3 + Y_6)$$
$$e_2 = (\sqrt{3}/6)\,(Y_1 - Y_2 + Y_4 - Y_5)$$
$$E_3 = (1/6)\,(-Y_1 + Y_2 - Y_3 + Y_4 - Y_5 + Y_6)$$

EIGHT-POINT SCHEDULE—for d-c, 1st, 2nd, 3rd, and 4th harmonics

$$E_0 = (1/8)\,(Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8)$$
$$E_1 = (\sqrt{2}/8)\,(Y_1 - Y_3 - Y_5 + Y_7) + (1/4)\,(-Y_4 + Y_8)$$
$$e_1 = (\sqrt{2}/8)\,(Y_1 + Y_3 - Y_5 - Y_7) + (1/4)\,(Y_2 - Y_6)$$
$$E_2 = (1/4)\,(-Y_2 + Y_4 - Y_6 + Y_8)$$
$$e_2 = (1/4)\,(Y_1 - Y_3 + Y_5 - Y_7)$$
$$E_3 = (\sqrt{2}/8)\,(-Y_1 + Y_3 + Y_5 - Y_7) + (1/4)\,(-Y_4 + Y_8)$$
$$e_3 = (\sqrt{2}/8)\,(-Y_1 + Y_3 - Y_5 - Y_7) + (1/4)\,(-Y_2 + Y_6)$$
$$E_4 = (1/8)\,(-Y_1 + Y_2 - Y_3 + Y_4 - Y_5 + Y_6 - Y_7 + Y_8)$$

TWELVE-POINT SCHEDULE—for d-c, 1st, 2nd, 3rd, 4th, 5th, and 6th harmonics

$$E_0 = (1/12)\,(Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8 + Y_9 + Y_{10} + Y_{11} + Y_{12})$$
$$E_1 = (\sqrt{3}/12)\,(Y_1 - Y_5 - Y_7 + Y_{11}) + (1/12)\,(Y_2 - Y_4 - Y_8 + Y_{10}) + (1/6)\,(-Y_6 + Y_{12})$$
$$e_1 = (1/12)\,(Y_1 + Y_5 - Y_7 - Y_{11}) + (\sqrt{3}/12)\,(Y_2 + Y_4 - Y_8 - Y_{10}) + (1/6)\,(Y_3 - Y_9)$$
$$E_2 = (1/12)\,(Y_1 - Y_2 - Y_4 + Y_5 + Y_7 - Y_8 - Y_{10} + Y_{11}) + (1/6)\,(-Y_3 + Y_6 - Y_9 + Y_{12})$$
$$e_2 = (\sqrt{3}/12)\,(Y_1 + Y_2 - Y_4 - Y_5 + Y_7 + Y_8 - Y_{10} - Y_{11})$$
$$E_3 = (1/6)\,(-Y_2 + Y_4 - Y_6 + Y_8 - Y_{10} + Y_{12})$$
$$e_3 = (1/6)\,(Y_1 - Y_3 + Y_5 - Y_7 + Y_9 - Y_{11})$$
$$E_4 = (1/12)\,(-Y_1 - Y_2 - Y_4 - Y_5 - Y_7 - Y_8 - Y_{10} - Y_{11}) + (1/6)\,(Y_3 + Y_6 + Y_9 + Y_{12})$$
$$e_4 = (\sqrt{3}/12)\,(Y_1 - Y_2 + Y_4 - Y_5 + Y_7 - Y_8 + Y_{10} - Y_{11})$$
$$E_5 = (\sqrt{3}/12)\,(-Y_1 + Y_5 + Y_7 - Y_{11}) + (1/12)\,(Y_2 - Y_4 - Y_8 + Y_{10}) + (1/6)\,(-Y_6 + Y_{12})$$
$$e_5 = (1/12)\,(Y_1 + Y_5 - Y_7 - Y_{11}) + (\sqrt{3}/12)\,(-Y_2 - Y_4 + Y_8 + Y_{10}) + (1/6)\,(Y_3 - Y_9)$$
$$E_6 = (1/12)\,(-Y_1 + Y_2 - Y_3 + Y_4 - Y_5 + Y_6 - Y_7 + Y_8 - Y_9 + Y_{10} - Y_{11} + Y_{12})$$

SIGNIFICANCE OF NOTATIONS

D-c component $= E_0$

Fundamental a-c component $= e_1 + jE_1$; peak value $= \sqrt{e_1^2 + E_1^2}$

Second harmonic a-c component $= e_2 + jE_2$; peak value $= \sqrt{e_2^2 + E_2^2}$

Third harmonic a-c component $= e_3 + jE_3$; peak value $= \sqrt{e_3^2 + E_3^2}$

Fourth harmonic a-c component $= e_4 + jE_4$; peak value $= \sqrt{e_4^2 + E_4^2}$

Fifth harmonic a-c component $= e_5 + jE_5$; peak value $= \sqrt{e_5^2 + E_5^2}$

Sixth harmonic a-c component $= E_6$ (peak value)

The phase angles of the various a-c components with the time base of the complex wave are generally of no interest, but may be obtained from the complex notations if desired.

Example 1

To illustrate the application of the formulas just developed for a six-point analysis, let us analyze the complex wave of Fig. 3, whose period is 1/10 sec (frequency 10 cycles).

First, divide the time period of one cycle into six equal intervals, and measure the instantaneous values of the complex wave at these times, tabulating as follows:

Points	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Values	2	4	1	−1	−2	1

Substituting these values in the six-point formulas in Table I, we obtain

$E_0 = (1/6)\ (2 + 4 + 1 - 1 - 2 + 1) = 0.833$
$E_1 = (1/6)\ (2 - 4 + 1 - 2) + (1/3)\ (-1 + 1) = -0.5$
$e_1 = (\sqrt{3}/6)\ (2 + 4 + 1 + 2) = 2.6$
$E_2 = (1/6)\ (-2 - 4 + 1 + 2) + (1/3)\ (1 + 1) = 0.167$
$e_2 = (\sqrt{3}/6)\ (2 - 4 - 1 + 2) = -0.289$
$E_3 = (1/6)\ (-2 + 4 - 1 - 1 + 2 + 1) = 0.5$

The results of the analysis are

D-c component $= E_0 = 0.833$ volt
Fundamental a-c component (10 cycles) $= 2.6 - j\,0.5 = 2.65$ volts (peak)
Second harmonic a-c component (20 cycles) $= -0.289 + j\,0.167 = 0.333$ volt (peak)
Third harmonic a-c component (30 cycles) $= E_3 = 0.5$ volt (peak)

Example 2

As a further illustration of the method, let us apply the twelve-point analysis to the output waveform of a 60-cycle half-wave rectifier output having a peak value of 100 volts. The graph of this wave form is shown in Fig. 4.

Note that the period of a complete cycle of this particular wave extends along the time base from $t = 0$ to $t = 1/60$ sec. Divide this time of one complete cycle into twelve equal intervals (1/720 sec), measure the instantaneous values of the wave at these times, and tabulate as follows (in this case the values may be calculated inasmuch as the wave is exactly half a sine wave whose equation is known)

Points	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6–Y_{12}
Values	50	86.6	100	86.6	50	0

Substituting these values into the twelve-point formulas of Table I gives

$E_0 = (1/12)\ (50 + 86.6 + 100 + 86.6 + 50) = 31.1$
$E_1 = (\sqrt{3}/12)\ (50 - 50) + (1/12) \times (86.6 - 86.6) = 0$
$e_1 = (1/12)\ (50 + 50) + (\sqrt{3}/12) \times (86.6 + 86.6) + (1/6)\ (100) = 50.0$
$E_2 = (1/12)\ (50 - 86.6 - 86.6 + 50) + (1/6)\ (-100) = -22.8$
$e_2 = (\sqrt{3}/12)\ (50 + 86.6 - 86.6 - 50) = 0$
$E_3 = (1/6)\ (-86.6 + 86.6) = 0$
$e_3 = (1/6)\ (50 - 100 + 50) = 0$
$E_4 = (1/12)\ (-50 - 86.6 - 86.6 - 50) + (1/6)\ (100) = -6.09$
$e_4 = (\sqrt{3}/12)\ (50 - 86.6 + 86.6 - 50) = 0$
$E_5 = (\sqrt{3}/12)\ (-50 + 50) + (1/12) \times (86.6 - 86.6) = 0$
$e_5 = (1/12)\ (50 + 50) + (\sqrt{3}/12) \times (-86.6 - 86.6) + (1/6)\ (100) = 0$
$E_6 = (1/12)\ (-50 + 86.6 - 100 + 86.6 - 50) = -2.23$

As in the previous example, these results simplify to

D-c component $= E_0 = 31.1$ volts
Fundamental a-c component $= 50.0$ volts (60 cycles)
Second harmonic component $= 22.8$ volts
Third harmonic component $= 0$
Fourth harmonic component $= 6.1$ volts
Fifth harmonic component $= 0$
Sixth harmonic component $= 2.23$ volts

The Fourier analysis for this same wave shows the respective values of the components to be 31.8, 50.0, 21.2, 4.24 and 1.82 volts. A comparison with the values obtained with the graphical method above shows good correlation. It should be noted here that the Fourier analysis can only be applied when the equation of the wave is known or can be pieced together, while the graphical method may be applied in all cases.

The following rules and suggestions will prove of value in applying the graphical analysis.

1. Be certain to take a full cycle of the complex wave to apply the analysis. The choice of the starting and end points is not important so long as they cover the interval of an entire cycle.

2. All values of a-c components in the analysis are peak values.

3. In certain cases, where a sort of symmetry exists in the wave, the d-c and/or certain of the a-c components may be zero.

4. Frequently, we are not interested in absolute values of the components, but relative ones with respect to the complex wave. In these cases, any arbitrary scale of values may be assigned to the complex wave and the values of the components expressed as percentages.

5. The values of the components of a given wave obtained by the 6, 8 or 12-point method may differ somewhat, becoming more accurate as a greater number of points are taken. In a sense, we may look on the Fourier analysis as one taking an infinite number of points.

6. Should the phase angle of any of the a-c components be desired, it will be given as arctan e/E. Further, it should be noted that one cycle of the complex wave represents 360° at the fundamental frequency, 720° at the second harmonic, etc.

7. The usual care in the observation of algebraic signs should be taken when measuring the Y values and substituting them into the formulas.

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(2) Denman, R. P. G., 36 and 72 Ordinate Schedules for General Harmonic Analysis, ELECTRONICS, Sept., 1942, p. 44. Systematization of procedure facilitates the tedious calculations required in harmonic analysis of complex waves. The schedules, with equations for checking results, are applicable to use with odd and even harmonics.

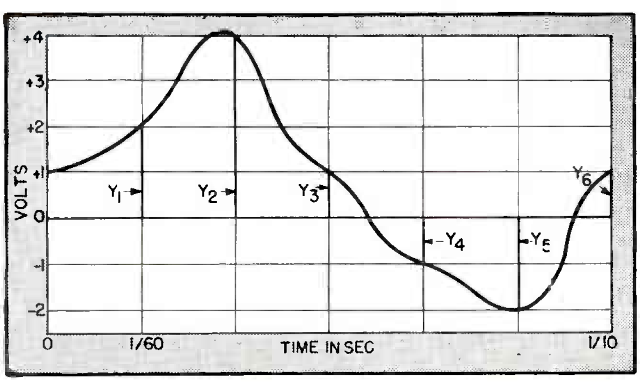

FIG. 3—Complex wave analyzed in Example 1 by means of the six-point schedule in Table I

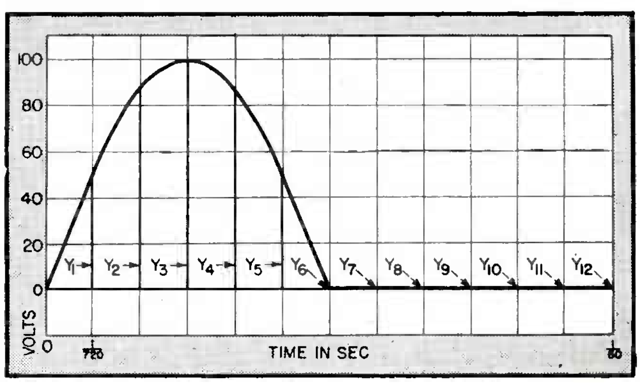

FIG. 4—Output wave form of half-wave rectifier, analyzed in Example 2 by means of the twelve-point schedule