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SOME GENERAL PROPERTIES OF THE EXACT ACOUSTIC FIELDS IN HORNS AND BAFFLES

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The propagation of the fundamental, longitudinal acoustic mode in a duct of variable cross-section is considered, and the "Webster" wave equations for the sound pressure and velocity are used to establish some general properties of the exact acoustic fields. The equipartition of kinetic and compression energies is shown (section 2.1) to hold at all stations only for (i) a duct of constant cross-section and (ii) an exponential horn; these are the two cases for which the wave equations for the acoustic velocity and pressure coincide. It is proved (section 2.3) that there are only five duct shapes, forming two dual families, which have constant cut-off frequency(ies): namely, (I) the exponential duct, which is self-dual, and is the only shape with constant (and coincident) cut-offs both for the velocity and pressure; (II) the catenoidal horns, of cross-section $S \sim \cosh^2$, $\sinh^2$, which, with their duals (III) the inverse catenoidal ducts $S \sim \mathrm{sech}^2$, csch^2, have one constant cut-off frequency, respectively, for the acoustic pressure and velocity. The existence of at least one constant cut-off frequency implies that the corresponding wave equation can be transformed into one with constant coefficients, and thus the acoustic fields calculated exactly in terms of elementary (exponential, circular and hyperbolic) functions; this property also applies to the imaginary transformations of the above shapes, viz., the sinusoidal $S \sim \sin^2$ and inverse sinusoidal $S \sim \csc^2$ ducts, that have no cut-off frequency, i.e., are acoustically "transparent". It is shown that elementary exact solutions of the Webster equation exist only (section 3.1) for these seven shapes: namely, the exponential, catenoidal, sinusoidal and inverse ducts; it is implied that for all other duct shapes the exact acoustic fields involve special functions, in infinite or finite terms, e.g., Bessel and Hermite functions respectively for power-law and Gaussian horns. Examples of the method of analysis are given by calculating, in elementary form, the exact acoustic fields in inverse catenoidal ducts, for all cases of (a) propagating waves above, (b) non-oscillating modes below and (c) transition fields at the cut-off frequency. The inverse catenoidal ducts consist of (A) the horn of cross-section $S \sim \mathrm{sech}^2$, ressembling the "soliton" of non-linear water wave fame, and (B) the baffle of cross-section $S \sim \mathrm{csch}^2$, which also matches two exponentially converging ducts, but has infinite, instead of finite, flare at the origin. The geometrical and acoustic properties of these ducts are illustrated by sets of six plots, in Figure 1(a) for the sech-horn and in Figure 1(b) for the csch-baffle; the exact acoustic fields are described by amplitude and phase decompositions of the sound velocity and pressure, plotted as functions of position along the duct, for four frequencies ranging from the cut-off condition to the ray limit (or W.K.B.J. approximation).

1. INTRODUCTION

The consideration of the acoustics of ducts and analogous problems (section 1.1), and the indication of the practical applications motivating their study (section 1.2), raises a number of general questions (section 1.3), which are the subject of the present paper.

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1.1. ANALOGIES IN THE MECHANICS AND ELECTRODYNAMICS OF CONTINUA

The wave equation describing the one-dimensional, longitudinal propagation of sound in a duct of variable cross-section (horn) is similar to that governing the transversal vibrations of a flexible string of non-uniform thickness. The latter was considered first [1, J. Bernoulli 1753; 2, Euler 1764; 3, D. Bernoulli 1765; 4, Euler 1766] in the context of linear elasticity, and the analogy with the acoustics of gases in non-uniform tubes was quickly recognized [5, Lagrange 1760; 6, Euler 1771]. Outstanding features of these early researches were the recognition of aerial acoustics as an important branch of fluid mechanics [7, Euler 1755], and the successful use of physical acoustics to explain musical sound [8, Euler 1818]. The detailed evolution of the subject in the XVIIIth century, including the early errors and controversies, is an interesting topic of the history of science [9, Truesdell 1955; 10, Truesdell 1960]. The equation of acoustic horns was rediscovered independently much later [11, Rayleigh 1916; 12, Webster 1919], and thus its current designation as "Webster's equation" is misleading as concerns the history of the subject, although it was only in this century that the topic has been once more widely and intensively researched.

The literature on this subject is more sparse in the XIXth century, but significant results were demonstrated, including the analogy between the two preceding problems and water waves in a channel of varying width [13, Green 1836] and electric currents in non-uniform transmission lines [14, Heaviside 1882]. In connexion with the former it was shown that, if the channel tapers gradually, the wave amplitude varies with the inverse square root of the cross-section (or width, for uniform depth); this result implies neglecting the reflections due to the walls [15, Rayleigh 1894, see Volume 2 p. 68], and corresponds to the modern Wentzel–Kramers–Brillouin–Jeffreys or W.K.B.J. approximation [16, Brekhovskikh 1960]. The exact solutions for the acoustic fields in ducts were considered in detail for the simpler shapes, such as the cone and cylinder [17, Duhamel 1839; 18, Pochhammer 1876], which appear to be subjects of almost perennial research [19, Barton 1908; 20, Hoersch 1925]. Other analogies, besides the acoustic horn, non-uniform string, channel with varying width and non-homogeneous transmission line, include [21, Eisner 1966] torsional waves in bars, the "water hammer" in hydraulics and the solid horns used as displacement amplifiers.

The rediscovery of the horn equation [12, Webster 1919] was followed by considerable research concerning the impedance and resonance characteristics of various duct shapes, both old and new. For the duct of cross-section $S \sim x^n$ [5, Lagrange 1760], the acoustic fields can be expressed exactly in terms of Bessel functions [22, Ballantine 1927], justifying the designation "Bessel horns". These include, for $n = 2$, the conical horn [23, Stewart 1920] containing a spherical wave [24, Euler 1759], and for $n = 1$, the parabolic horn [25, Olson and Wolff 1930]; the limit $n \to \infty$ leads to the exponential horn [26, Hanna 1927], which has been studied for sustained, small-amplitude waves [27, Hanna and Slepian 1924], and for finite amplitude [28, Goldstein and McLachlan 1934] and transient [29, McLachlan and McKay 1936] effects. Other shapes for which exact solutions have been obtained include the Gaussian [30, Parodi 1945], hypex [31, Salmon 1946], catenoidal [32, Thiessen 1950], tractrix [33, Lambert 1954] and sinusoidal [34, Nagarkar and Finch 1971] horns. By transformation of these [35, Mawardi 1949; 36, Pinkney and Basso 1963; 37, Pyle 1965; 38, Molloy 1975] additional exact solutions have been obtained. The matching of different ducts [39, Poisson 1817] has also been considered, and is a means [40, Olson 1938; 41, Merkulov and Kharitonov 1959] whereby desirable impedance characteristics can be obtained over a wider range of frequencies than would be possible by using a single, simple shape.

1.2. APPLICATIONS TO VOICE, MUSIC AND SOUND

One of the motivations for the study of the acoustics of horns in this century has been the design of loudspeakers in particular [42, Crandall 1927; 43–45, McLachlan 1934, 1935, 1936; 46, Jordan 1963], and high-quality sound reproduction in general [47, Olson and Massa; 48, Olson 1940; 49, Moir 1961; 50, Olson 1972]. The use of solid horns as displacement amplifiers [51, Eisner 1963] and applications of electromagnetic horns [52, Stevenson 1952] have been additional motivations. Other analogous problems, such as the vibrations of tapering bars [53, Bies 1962], the design of ultrasonic concentrators [54, Merkulov 1957] and the analysis of non-uniform transmission lines [55, Schwartz 1964], have important applications in mechanical and electrical engineering. The realization that some horn shapes were discovered empirically before being analyzed theoretically, e.g., the sinusoidal duct [34, Nagarkar and Finch 1971] is used in the mouth of the English horn, and the Bessel ducts [22, Ballantine 1927] are used, with various exponents, in instruments of the brass family [56, Benade 1976], emphasizes the connexion between horns and musical acoustics [57, Jeans 1937; 58, Benade 1980; 59, Berg and Storck 1982]. The acoustics of ducts is also relevant to hearing and speech, e.g., to sound transmission in the outer ear and to sound formation in the vocal tract [60, Schroeder 1967; 61, Mermelstein 1967; 62, Ishikawa, Matsudaira and Kaneko 1976; 63, Jackson, Butler and Pyle 1978]. An important generalization of the acoustics of horns is the case of ducts carrying a mean flow, which has applications in aircraft nozzles [64, Nayfeh, Kaiser and Telionis 1975].

The basic problem of sound propagation in ducts of variable cross-section, either without (horns) or with (nozzles) mean flow, has several ramifications, to which only passing reference is appropriate here: (i) the analogy between the acoustics of solid, hard-walled frictionless nozzles and abstract "ray tubes" in a free flow [65, 66, Campos 1978]; (ii) alternative derivations of "Webster's" equation by using variational methods [67, Weibel 1955] or a limit to the three-dimensional case [68, Stevenson 1955]; (iii) three-dimensional solutions [69, Pyle 1967], exact for certain shapes, e.g., the hyperbolic horn without [70, Freehafer 1940] and with [71, Cho 1980] baffle; (iv) effects of resonances and radiative losses [72, Benade and Jansson 1965; 73, Jansson and Benade 1965], and of evanescent modes and visco-thermal losses at the walls [74, 75, Kergomard 1981, 1982]; (v) generalizations to non-uniform media, either elastic [76, Shaw 1970] or fluid [77, Bergmann 1946], including exact solutions in strong stratification [78, 79, Campos 1983]; (vi) cases of curved ducts [80, Cabelli 1980] and ducts whose walls are elastic [81, Sinai 1981], have non-uniform impedance [82, Namba and Fukushige 1980] or are lined regularly [83, Yoshida 1981] or randomly [84, Howe 1983]; (vii) the effect of mean flow in nozzles on the generation [85, Morfey 1971; 86, Davies 1981], propagation [87, Perulli 1978; 88, Mani 1981] and energy flux [89, 90, Mohring 1971, 1973]; (viii) the use of approximate solutions, e.g., the W.K.B.J. or ray limit [91, Salmon 1946], conical steps [92, Zamorski and Wyrzykowski 1981], perturbation [93, Nayfeh, Kaiser and Telionis 1975] and wave envelope [94, Kaiser and Nayfeh 1977] techniques; (ix) comparisons of theory and experiment for horns and nozzles [95, Plumblee, Dean, Wynne and Burrin 1973; 96, Nayfeh, Kaiser, Marshall and Hurst 1981; 97, Wilcox and Lester 1982]; (x) the use of numerical [98, Mohring and Raman 1976; 99, Bostrom 1983] and analytic [100, 101, Campos 1983] methods in the acoustics of non-uniform flows in exhausts and nozzles.

1.3. PARTITION OF ENERGY AND FILTERING PROPERTIES

The basic problem in the acoustics of ducts is to determine the wave fields [102, Landau and Lifshitz 1953; 103, Morse and Ingard 1968], and from these can be calculated the impedance, and energy densities and fluxes, which are relevant in applications [104, Kinsler and Frey 1950; 105, Olson 1967]. It is well-known that there is equipartition of kinetic and compression energies for a plane wave, corresponding to propagation in a duct of constant cross-section; by way of contrast, one should note that for a conical horn, corresponding to a spherical wave, there is *no* equipartition of kinetic and compression energies, except asymptotically. The question thus arises of which duct shapes are consistent with equipartition of kinetic and compression energies at all positions. It can be shown that, besides the uniform duct, only the exponential duct has this property (section 2.1); the reason is that, in order for the initial equipartition of energy to hold throughout the duct, the acoustic pressure and velocity must evolve in the same way, i.e., satisfy identical wave equations, and this is only true for the uniform and exponential ducts. This result is obtained in the context of one-dimensional propagation [106, Levine 1978] in ducts of varying cross-section; the validity of the one-dimensional assumption has been discussed in detail in the literature (see, e.g., reference [43]), and it is sufficient to note that it excludes transversal modes and does not apply to high-frequency waves near horn lips, but is adequate to represent the fundamental, longitudinal mode in ducts of moderate flare.

As mentioned, there is only *one* non-uniform duct which preserves equipartition of kinetic and compression energies at all stations, namely, the exponential duct; the latter is well-known [107, Lighthill 1978; 108, Dowling and Ffowcs Williams 1983] to have a constant cut-off frequency; this cut-off applies both to the acoustic pressure and velocity, since they satisfy identical equations in this, and *only* in this, shape of horn. The catenoidal horns, of cross-section $S \sim \cosh^2$, $\sinh^2$ are known [31, 32] to have a constant cut-off for the acoustic pressure, and therefore, the "dual" shapes [37], viz., the inverse catenoidal ducts $S \sim \operatorname{sech}^2$, csch^2, have a constant cut-off for the acoustic velocity. The question of whether other duct shapes exist which have a constant cut-off frequency may be answered (section 2.3) in the negative. Thus there are only *five* duct shapes with constant cut-off frequencies, namely the exponential, catenoidal and inverse catenoidal. The latter two have received less attention in the literature than the former three, and it may be worth noting that the inverse catenoidal ducts are (i) the sech^2-horn, whose shape coincides with that of a "solitary wave" known in the context of non-linear water waves [109, Whitham 1974], whereas (ii) the csch^2-baffle coincides with the sech^2-horn in its asymptotic convergence, but contrasts in having an infinitely flaring mouth instead of a finite maximum cross-section. A consequence of the existence of a constant cut-off frequency is that the acoustic fields can be expressed exactly in terms of elementary (exponential, circular and hyperbolic) functions; this property is shared with the sinusoidal $S \sim \sin^2$ and inverse sinusoidal $S \sim \csc^2$ ducts, since the latter can be obtained from their catenoidal counterparts [34] by means of an imaginary change of variable, which also eliminates the cut-off frequency: i.e., there is no filtering. Thus the exact acoustic fields in ducts of non-uniform cross-section can be expressed in terms of elementary functions in the case of *seven* shapes: namely, the exponential, catenoidal, sinusoidal and inverse ducts; furthermore, it can be proved (section 3.1) that the preceding list is exhaustive. The implication is that, for all other duct shapes, the exact expression of the acoustic fields requires the use of special functions, in infinite or finite terms. For example, the acoustic fields in the ducts of power-law shape $S \sim x^{2n}$ can be expressed in terms of Bessel functions of order $n-1/2$ [22], which can be expressed in finite terms in the case when n is an integer (spherical Bessel functions), e.g., for the conical horn $n=1$; in a somewhat analogous manner, the acoustic fields in the Gaussian horn are given by Hermite functions [53], which reduce to Hermite polynomials for a discrete set of boundary conditions at the ends.

With the motivations for the present work thus indicated, the introduction can be concluded by outlining the method of analysis which leads to the proof of the general properties concerning equipartition of kinetic and compression energies, existence of constant cut-off frequencies, and determination of exact elementary solutions. In section 2.1 the wave equations satisfied by the acoustic pressure and velocity in horns are deduced, and it is shown that they are generally different, and only coincide (equipartition of kinetic and compression energies) in ducts of varying cross-section for the exponential shape; this horn has constant (and coincident) cut-off frequencies for the acoustic pressure and velocity, and it is shown that four other shapes exist with *one* cut-off (section 2.2), viz., the catenoidal (cosh, sinh)/inverse catenoidal (sech, csch) ducts have a constant cut-off (section 2.3), respectively, for the sound pressure/velocity; in section 3 it is shown that the exact acoustic fields can be expressed in terms of elementary functions not only for these five duct shapes but also for the sinusoidal and inverse sinusoidal ducts (section 3.1), the resulting list being exhaustive; as an example, the wave equation for the acoustic velocity is transformed into one with constant coefficients and cut-off (section 3.2), in the case of the inverse catenoidal ducts, namely (section 3.3), the sech-horn and csch-baffle; in section 4, for the latter two shapes, the exact acoustic fields are given, for propagating waves above (section 4.1), non-oscillating modes below (section 4.3) and transition fields at (section 4.2) the cut-off frequency; the initial conditions specify the pressure and velocity, or a derivative of the velocity which is independent of the initial pressure, e.g., the dilatation or rate of dilatation; in section 5 the discussion of basic geometrical and acoustic properties (section 5.2) is illustrated by dimensionless plots (section 5.1) concerning the sech-horn (Figure 1(a)) and csch-baffle (Figure 1(b)); the quantities plotted as functions of axial co-ordinate, and interpreted physically are (i) the cross-section, radius and length scale, (ii) the amplitude of the acoustic velocity and pressure, with (iii) the latter decomposed into primary and secondary sound fields (respectively in- and out-of-phase to the velocity), and (iv) the phase difference between pressure and velocity, with the waveforms given for four frequencies ranging from the cut-off to the ray limit.

2. ACOUSTIC PRESSURE AND VELOCITY IN HORNS

In this section the propagation of the fundamental longitudinal acoustic mode in horns is considered, and the acoustic pressure and velocity are compared as regards the wave equations (section 2.1) they satisfy, the resulting conditions for wave propagation (section 2.2), and the duct shapes for which the cut-off frequency is constant (section 2.3).

2.1. CASES OF EQUIPARTITION OF KINETIC AND COMPRESSION ENERGIES

The problem to be considered is that of the acoustics of a horn, i.e., a duct of varying cross-section, for the fundamental (or lowest order) longitudinal mode. The latter corresponds to the wave variables being uniform over the cross-section, and depending only on the co-ordinate z along the axis and on the time t. The equation of continuity expresses the conservation of mass per unit length ρS, where ρ is the mass density and S the cross-section (a list of symbols is given in the Appendix),

$$\partial(\rho S)/\partial t + \partial(\rho S v)/\partial z = 0, \tag{1a}$$

where v is the velocity; the equation of momentum is a balance of total (local plus convective) acceleration against the gradient of pressure p:

$$\partial v/\partial t + v\, \partial v/\partial z + \rho^{-1}\, \partial p/\partial z = 0. \tag{1b}$$

The acoustic problem is to be considered under the following conditions: (i) the flared horn has hard, non-distensible walls, so that the cross-section $S(z)$ depends only on the axial distance z; (ii) the mean state of the fluid is of rest $v_0=0$, with constant density ρ and pressure p_0; (iii) for small amplitude waves the equations are linearized in the velocity v, pressure p and density ρ' perturbations, which depend on space z and time t; (iv) the propagation is adiabatic, so that the sound speed $c^2 \equiv (\partial p_0/\partial \rho)_s$ relates pressure p and density ρ' perturbations and gradients. The linearized continuity (1a) and momentum (1b) equations read

$$\partial p/\partial t + (\rho c^2/S)\, \partial(Sv)/\partial z = 0, \qquad \partial v/\partial t + \rho^{-1}\, \partial p/\partial z = 0, \tag{2a, b}$$

in terms of the acoustic velocity v and pressure p, with ρ denoting the constant, mean mass density.

Eliminating between equations (2a, b) gives the wave equations

$$S^{-1}\partial\{S(\partial p/\partial z)\}/\partial z - c^{-2}\partial^2 p/\partial t^2 = 0, \tag{3a}$$

$$\partial\{S^{-1}\{\partial(Sv)/\partial z\}/\partial z - c^{-2}\partial^2 v/\partial t^2 = 0, \tag{3b}$$

respectively, for the acoustic pressure and velocity. Both wave equations reduce to the classical form in the case of a plane wave in a uniform duct, but they are generally different in horns (of variable cross-section). The conditions of validity of the one-dimensional approximation leading to the wave equations (3a, b) have been discussed in the literature [43, 64, 65]; comparison of the wave equations for the pressure p (3a) and velocity v (3b) demonstrates [37] the duality principle,

$$S \leftrightarrow 1/S, \qquad p \leftrightarrow Sv, \qquad Z \equiv p/Sv \leftrightarrow Sv/p \equiv 1/Z, \tag{3c}$$

stating that if the cross-section S is inverted to yield the "dual horn" $1/S$, the pressure and the volume velocity Sv (equal to the velocity times the cross-section) are interchanged; from the first and second transformations in expressions (3c) follows the third, stating that dual ducts have inverse impedances: i.e., the impedance of a duct is the admittance of its dual.

Since the horn is assumed to be rigid, the coefficients of the wave equations are time independent, and it is convenient to use a Fourier decomposition in time,

$$v, p(z, t) = \int_{-\infty}^{+\infty} V, P(z; \omega)\, e^{-i\omega t}\, d\omega, \tag{4a, b}$$

where $v, p(z, t)$ are the acoustic velocity and pressure perturbations at position z and time t, and $V, P(z; \omega)$ are the velocity and pressure perturbation spectra at station z for a wave of frequency ω. In situations where the acoustic fields v, p and their spectra V, P either cannot be confused, or have the same properties, one may use simply the brief expression (acoustic) pressure, velocity; otherwise the distinction is made. The acoustic fields satisfy equations (3a, b), and thus one obtains for their spectra

$$P'' + (S'/S)P' + (\omega/c)^2 P = 0, \tag{5a}$$

$$V'' + (S'/S)V' + \{(\omega/c)^2 + (S'/S)'\}V = 0, \tag{5b}$$

where a prime denotes a derivative with regard to z. The wave equations for the pressure (5a) and velocity (5b) coincide only in the case $(S'/S)'=0$, i.e., $S'/S = 1/L \equiv$ constant, corresponding to a horn of exponential cross-section $S(z) = S(0)\, e^{z/L}$; this is the only case, besides the plane wave in a uniform duct, for which the ratio of pressure to velocity is independent of position ($P(z; \omega)/V(z; \omega)$ = function of ω alone), and equipartition of compression and kinetic energies holds throughout the horn. For any horn of non-exponential shape the ratio of acoustic pressure to velocity varies along its length, and thus the initial equipartition of kinetic and compression energies is generally *not* preserved (an example is given in section 3.3).

2.2. REDUCED VARIABLES AND ACOUSTIC WAVE INVARIANTS

In order to obtain the acoustic fields in a horn it is sufficient to solve one of the wave equations (5a, b), and then use (as derived from equations (2a, b) and (4a, b)) one of the polarization relations

$$P(z;\omega) = -\mathrm{i}(\rho c^2/\omega)\{S(z)\}^{-1}\,\mathrm{d}\{S(z)V(z;\omega)\}/\mathrm{d}z, \tag{6a}$$

$$V(z;\omega) = -(\mathrm{i}/\rho\omega)\,\mathrm{d}P(z;\omega)/\mathrm{d}z: \tag{6b}$$

i.e., one can either (a) solve equation (5a) for the pressure and use equation (6b) to determine the velocity, or (b) solve equation (5b) for the velocity and use equation (6a) to determine the pressure. This procedure has the advantage that it involves only two constants of integration, as appropriate to a second-order problem; if the two wave equations (5a, b) were solved, four constants of integration would result, and one of the polarization relations (6a, b) would still be needed to express them in terms of two (the other polarization relation should then be satisfied identically). Although the wave equation for the pressure (5a) appears, on first inspection, simpler than that for the velocity (5b), there are instances where the latter proves more amenable to simplification than the former (an example is given in section 3.2); thus it is appropriate to proceed by considering in parallel the wave equations for the pressure (5a) and velocity (5b), so as to retain the option of using whichever is simpler to integrate for a given, particular duct shape.

In the W.K.B.J. approximation or ray limit, which is valid for high-frequencies over short distances, the cross-section tapers slowly on the scale of a wavelength, and the amplitude varies like the inverse square root of the cross-section [13, 15, 16], this suggests the change of variable

$$P,\ V(z;\omega) = \{S(z)\}^{-1/2}\{Q,\ W(z;\omega)\}, \tag{7a, b}$$

so that the reduced pressure Q and velocity W, unlike the acoustic pressure P and velocity V, have a constant amplitude initially, in the high-frequency limit. The change of variable (7a, b) is precisely that needed to omit the term involving the first derivative [110, Forsyth 1929] in the linear, second-order differential equations, transforming them to the "Schrodinger" equation [30, 36], or standard, form,

$$Q'' + J_p Q = 0, \qquad W'' + J_v W = 0, \tag{8a, b}$$

where the wave invariants for the pressure J_p and velocity J_v are given respectively by

$$J_p \equiv \omega^2/c^2 - (1/2)(S''/S) + (1/4)(S'/S)^2, \tag{9a}$$

$$J_v \equiv \omega^2/c^2 + (1/2)(S''/S) - (3/4)(S'/S)^2; \tag{9b}$$

these reduce to a positive constant $J_p, J_v \sim k^2$, namely, the square of the wavenumber $k \equiv \omega/c$, in the W.K.B.J. limit, confirming that, in the latter case, the reduced pressure Q (8a) and velocity W (8b), have constant amplitudes.

The wave invariants generally involve the cross-section $S(z)$ and its first two derivatives S', S'', whereas if one introduces the length scale L of cross-sectional variation, defined by

$$L \equiv S/S' = \{\mathrm{d}(\log S)/\mathrm{d}z\}^{-1}, \qquad S(z) = \exp\left\{\int^z \{L(\zeta)\}^{-1}\,\mathrm{d}\zeta\right\}, \tag{10a, b}$$

only the first derivatives appear in

$$J_p, J_v = \omega^2/c^2 - (1 \mp 2L')/(4L^2). \qquad (11a, b)$$

The interpretation of the wave invariants follows from expressions (8a, b), since (a) if they are negative, $J_p, J_v < 0$, only non-oscillating modes exist, (b) if they are positive, $J_p, J_v > 0$, wave propagation is possible. Thus the separation (c) of the two cases (a) and (b), namely, the vanishing of the wave invariants, $J_v, J_p = 0$, specifies via equations (11a, b) the cut-off frequencies ω_v, ω_p, respectively, for the acoustic velocity and pressure,

$$\omega_v, \omega_p = (c/2L)\{1 \pm 2L'\}^{1/2}, \qquad (12a, b)$$

separating the non-oscillating (below) and propagating (above) regions of the spectrum. The cut-off frequencies specify the properties of the horn as a high-pass filter of the acoustic velocity ω_v (12a) and pressure ω_p (12b).

2.3. HORNS AND BAFFLES WITH CONSTANT CUT-OFF FREQUENCY

From equations (12a, b) it is clear that the cut-off frequencies for the acoustic pressure and velocity coincide $\omega_p = \omega_v$ only in the case of constant length scale $L' = 0$: i.e., for the exponential horn,

$$\omega_p = \omega_v = c/2L, \qquad S(z) = S(0)\, e^{z/L}, \qquad (13a, b)$$

which has globally constant cut-offs. There may exist other horn shapes for which *one* of the cut-off frequencies ω_v, ω_p is a constant; in this case they can be written in the form

$$\omega_v, \omega_p = c/2l, \qquad l = L\{1 \pm 2L'\}^{-1/2}, \qquad (14a, b)$$

where l is a parameter with the dimensions of a length. It coincides with the length scale of cross-sectional variation $l = L$ only for the exponential horn $L' = 0$, which has constant and equal cut-offs for the velocity and pressure; a non-exponential horn, with a constant cut-off for the velocity/pressure, must have a non-uniform length scale $L(z)$ such that expression (14b) with, respectively, the $+/-$ sign reduces to a constant l.

The horns with a constant cut-off frequency $\omega = c/2l$ for the pressure/velocity, have cross-sections $S(z)$ satisfying the second-order, non-linear differential equations

$$2SS'' - S'^2 - S^2/l^2 = 0, \qquad 2SS'' - 3S'^2 + S^2/l^2 = 0, \qquad (15a, b)$$

obtained, respectively, by setting $J_p, J_v = 0$ in equations (9a, b). If instead of the cross-section $S(z)$ one uses the inverse of the length scale $L(z)$ defined by equation (10a), then equations (15a, b) reduce to non-linear first-order equations,

$$2(1/L)' \pm (1/L)^2 = \pm(1/l)^2, \qquad (16a, b)$$

where the upper/lower signs correspond to the cases of constant cut-off for the pressure/velocity, respectively. These equations could have been obtained from equations (14a, b), and are readily integrable; each has two solutions,

$$L(z) = l \begin{matrix}\coth\\ \tanh\end{matrix}(z/2l), -l \begin{matrix}\tanh\\ \coth\end{matrix}(z/2l), \qquad (17a, b)$$

for the length scale $L(z)$, which is positive/negative for the horns of constant cut-off respectively for the pressure/velocity. Substitution of equations (16a, b) in equation (10b) yields the cross-sections

$$S(z) = S_0 \begin{matrix}\cosh^2\\ \sinh^2\end{matrix}(z/2l), S_0 \begin{matrix}\operatorname{csch}^2\\ \operatorname{sech}^2\end{matrix}(z/2l), \qquad (18a, b)$$

showing, in agreement with equations (17a, b), that the ducts with constant pressure/velocity cut-off are, respectively, divergent/convergent away from the origin.

Thus one can make the following exhaustive listing of the horns with a constant cut-off frequency: (i) the *only* case of constant (and coincident) cut-off frequencies for both the acoustic pressure and velocity is the well-known exponential horn (13a, b); (ii) if only the cut-off for the pressure ω_p (12b) is required to be constant (14a), then, besides the exponential horn, the catenoidal horns, of cross-section $\cosh^2$, $\sinh^2$ (18a), also have this property; (iii) if only the cut-off frequency for the velocity ω_v (12a) is required to be constant (14a), then, besides the exponential horn, the inverse catenoidal ducts, of cross-sections sech^2, csch^2 (18b) also have this property. These five cases include two examples of dual horns [37, Pyle 1963], of (3c) inverse cross-sections $S \leftrightarrow 1/S$, and exchange of acoustic pressure p and volume velocity Sv: (a) the exponential horn is its own dual, and corresponds to the only case of two cut-offs, coincident for velocity and pressure; (b) the catenoidal $S \sim \cosh^2$, $\sinh^2$ and inverse catenoidal $1/S \sim \operatorname{sech}^2$, csch^2 ducts are duals, so that the constant cut-off for the acoustic pressure in the former [31, Salmon 1946] implies a constant cut-off for the velocity in the latter. It will be noted that all the four ducts with *one* constant cut-off are connexions of (I) exponential horns, which is the only case of *two* constant (and coincident) cut-offs: (A) the horns with constant pressure cut-off (18a) have a catenoidal profile (radius $R \sim \cosh$, $\sinh$ if they are axisymmetric), and match two exponential diverging horns $S(z) \sim (S_0/4)\,\mathrm{e}^{z/l}$ for $|z| \gg l$, through, respectively, (II) smooth and non-zero $S(z) \sim S_0(1 - z^2/4l^2)$ and (III) cusped and zero $S(z) \sim S_0(z^2/2l^2)$ sections for $z^2 \ll l^2$; (B) the cases of constant velocity cut-off (18b) correspond to the matching of exponential converging horns $S(z) \sim 4S_0\,\mathrm{e}^{-z/l}$ for $|z| \gg l$, through, respectively, (IV) a smooth, finite section $S(z) \sim S_0(1 + z^2/4l^2)$ for $z^2 \ll l^2$ for the horn of sech^2 cross-section (and sech radius), and (V) an infinite flare $S(z) \sim S_0(4l^2/z^2)$ at $z = 0$ for the baffle of csch^2 cross-section (or csch radius).

3. EXACT CALCULATION OF ACOUSTIC FIELDS

From the preceding analysis it follows that, whereas for the exponential horn it is immaterial which of the two wave equations is used, for the calculation of the acoustic fields in the catenoidal/inverse catenoidal ducts it is more convenient to start from the wave equation respectively for the pressure/velocity, since only this choice leads to the constant cut-off frequency. The existence of a constant cut-off frequency implies that the acoustic fields can be expressed exactly in terms of elementary functions for the exponential, catenoidal and inverse catenoidal ducts, and also for their imaginary transforms, viz., the sinusoidal and inverse sinusoidal ducts (section 3.1). After using the present theory to prove that this list of seven elementary exact solutions of Webster equation is exhaustive, the method can be illustrated in more detail in connexion with the inverse catenoidal ducts, namely the sech-horn and csch-baffle (sections 3.2 and 3.3).

3.1. EXISTENCE OF ELEMENTARY EXACT SOLUTIONS

By substituting expressions (12a, b) in equations (11a, b) the wave invariants can be written in the forms

$$J_v = (\omega^2 - \omega_v^2)/c^2, \qquad J_p = (\omega^2 - \omega_p^2)/c^2, \tag{19a, b}$$

confirming that they are positive above the cut-off and negative below, corresponding, from equations (8a, b) in the forms

$$Q'' + \{(\omega^2 - \omega_p^2)/c^2\}Q = 0, \qquad W'' + \{(\omega^2 - \omega_v^2)/c^2\}W = 0, \tag{20a, b}$$

respectively, to propagating $\omega > \omega_v, \omega_p$ and non-oscillating $\omega < \omega_v, \omega_p$ waves. In the case of constant cut-off frequency (14a), the wave equations for the reduced pressure Q and velocity W take the same simple form

$$\mathrm{d}^2\{Q, W(z;\omega)\}/\mathrm{d}(z/2l)^2 + \alpha\{Q, W(z;\omega)\} = 0, \qquad (21\mathrm{a, b})$$

with a constant, dimensionless coefficient,

$$\alpha \equiv 4\omega^2 l^2/c^2 - 1 = \omega^2/\omega_p^2 - 1, \qquad \omega^2/\omega_v^2 - 1, \qquad (22\mathrm{a, b})$$

replacing the wave invariants. Equation (21a) is typical of filtering processes, since $\alpha > 0$ for propagating waves, $\alpha < 0$ for non-oscillating modes and $\alpha = 0$ at the cut-off, and in all three cases the solution is elementary.

If one performs the imaginary transformation $z \to \mathrm{i}z$ (or $l \to -\mathrm{i}l$), the catenoidal (18a) and inverse (18b) ducts transform to the sinusoidal and inverse sinusoidal shapes,

$$S(z) = S_0 \begin{matrix}\sin^2\\ \cos^2\end{matrix}(z/2l),\ S_0 \begin{matrix}\csc^2\\ \sec^2\end{matrix}(z/2l), \qquad (23\mathrm{a, b})$$

where S_0 represents, respectively, the maximum/minimum cross-section (the cases $\cos^2$, $\sec^2$ can be omitted since they represent the same shape with a translation πl). The length scales (10a) corresponding to expressions (23a, b).

$$L(z) = \begin{matrix}+l\tan\\ -l\cot\end{matrix}(z/2l),\quad \begin{matrix}-l\tan\\ +l\cot\end{matrix}(z/2l), \qquad (24\mathrm{a, b})$$

lead (compare with expressions (17a, b)) to wave invariants (11a, b) which are always positive,

$$J_p = (\omega^2 + \omega_p^2)/c^2, \qquad J_v = (\omega^2 + \omega_v^2)/c^2, \qquad (25\mathrm{a, b})$$

in agreement with the imaginary transformation $l \to -\mathrm{i}l$, $\omega_{p,v} \to -\mathrm{i}\omega_{p,v}\omega_{p,v}^2 \to -\omega_{p,v}^2$ applied to equations (19a, b). It follows that the reduced pressure Q and velocity W from equations (8a, b), in the form

$$\mathrm{d}^2\{Q, W(z;\omega)\}/\mathrm{d}(z/2l)^2 + \beta\{Q, W(z;\omega)\} = 0, \qquad (26\mathrm{a, b})$$

always propagate, since the constant parameter

$$\beta \equiv 1 + 4\omega^2 l^2/c^2 = \omega^2 + \omega_v^2,\ \omega^2 + \omega_p^2, \qquad (27\mathrm{a, b})$$

is positive for all frequencies. The wave fields are also specified by elementary functions, but there is no cut-off frequency, and the sinusoidal [34, Nagarkar and Finch 1971] and inverse sinusoidal ducts are "acoustically transparent".

The catenoidal and sinusoidal ducts, and their duals, share with the exponential horn the property that the acoustic equation can be solved exactly in terms of elementary (exponential, sinusoidal and hyperbolic) functions. This can be the case only if the wave invariants J_p, J_v in expressions (8a, b) reduce to constants; the preceding cases have included all possible real values (9a, b) of J_p, J_v, namely, positive (19a, b) and (25a, b), negative and zero (19a, b), and thus the listing of elementary exact solutions of the horn equations is complete. Thus there are seven duct shapes for which the exact acoustic fields can be expressed in terms of elementary functions: (I) the exponential duct, which has two coincident cut-offs; (II, III) the catenoidal ducts which have a constant cut-off for the acoustic pressure (18a); (IV, V) the inverse catenoidal ducts, which have a constant cut-off for the velocity (18b); (VI, VII) the sinusoidal and inverse sinusoidal ducts, which have no cut-off, i.e., are acoustically "transparent" for the pressure/velocity, respectively. The two named last also satisfy the duality principle (3c), and in all seven cases the existence of elementary exact solutions is connected with the constant, global cut-off, and its imaginary transform.

3.2. GENERAL AND PARTICULAR METHODS OF TRANSFORMATION

The result that the exact acoustic fields in ducts can be expressed in terms of elementary functions only for shapes which are given by elementary functions appears self-evident. Its proof followed simply as a natural consequence of the method of analysis of the horn equations, by using the wave invariants (18a, b) and (9a, b). It may be worthwhile, as a comparison, to derive the elementary solution directly from the horn equations, to show that the process is rather less obvious, even with the benefit of hindsight. One can choose as examples the inverse catenoidal ducts, of cross-section (18b)

$$S(z) = S_0 \operatorname{sech}^2(z/2l), \qquad S_0 \operatorname{csch}^2(z/2l), \tag{28a, b}$$

where $S_0 = S(0)$ denotes the initial cross-section for the sech-horn (28a), and for the csch-baffle (28b), which has infinite area at the origin $S \to \infty$ as $z \to 0$, the constant $S_0 = S(z_0)$ denotes the area at the station $z_0 = 2l \sinh^{-1}(1) = 1{\cdot}76l$. The length scales (10a) corresponding (17b) to the cross-sections (28a, b) are given by

$$L(z) = -l \coth(z/2l), \; -l \tanh(z/2l), \tag{29a, b}$$

and have the following properties: (i) they are both negative at all stations z, since the cross-section decreases monotonically with the longitudinal co-ordinate z, and tend asymptotically to a constant value ($L \to -l$ as $z \to \infty$), since then the inverse catenoidal ducts reduce to convergent exponential horns, viz. $S(z) \sim 4S_0 \,\mathrm{e}^{-z/l}$ for $z^2 \gg l^2$; (ii) the absolute value of the length scale increases from the asymptotic value $|L(z)| > l$ for the sech-horn, and decreases $|L(z)| < l$ for the csch-baffle, since the former "flares-in" to a constant cross-section at the origin $S(z) \sim S_0(1 - z^2/4l^2)$ for $z^2 \ll l^2$, whereas the latter "flares out" in the line $z = 0$ according to the law $S(z) \sim S_0(2l/z)^2$ for $z^2 \ll l^2$.

The wave equations for the acoustic pressure (5a) and velocity (5b) are conveniently written by using the lengthscale L (10a) in the coefficients:

$$P'' + (1/L)P' + (\omega/c)^2 P = 0, \tag{30a}$$

$$V'' + (1/L)V' + \{(\omega/c)^2 + (1/L)'\}V = 0. \tag{30b}$$

In order to calculate the acoustic fields in the sech-horn (28a) and csch-baffle (28b) it is preferable to start from the equation for the acoustic velocity (30b), which should lead to a constant cut-off frequency for these shapes; equation (30a), which on first inspection appears simpler, would be less useful in attempting to derive the elementary solutions. One can introduce the dimensionless longitudinal co-ordinate

$$x \equiv z/2l, \qquad F(x) \equiv V(z;\omega), \tag{31a, b}$$

so that $F(x)$, which is the acoustic velocity perturbation spectrum in terms of x, satisfies (30b)

$$F'' - 2\begin{pmatrix} \tanh x \\ \coth x \end{pmatrix} F' + \begin{pmatrix} 1 + \alpha - 2\operatorname{sech}^2 x \\ 1 + \alpha + 2\operatorname{csch}^2 x \end{pmatrix} F = 0, \tag{32a, b}$$

where a prime denotes a derivative with regard to x, the constant factor α depends (22b) on the ratio of the wave frequency ω to the cut-off frequency ω_v, and the upper/lower coefficient corresponds, respectively, to the sech-horn (28a)/csch-baffle (28b).

The property of the sech-horn and csch-baffle, of having a constant cut-off frequency for the acoustic velocity, implies that it should be possible to transform the differential equation (32a, b) for the acoustic velocity perturbation spectrum into the standard type

$$G'' + \alpha G = 0, \tag{33}$$

where α depends only on frequency ω, and vanishes at the cut-off $\alpha(\omega_v) = 0$, in agreement with equation (22b). According to equation (7b), the transformation is effected by multiplying by the square root of the cross-section (28a, b) with (31a), viz., $\sqrt{S} \sim \text{sech}, \text{csch}\,(x)$, leading to the differential equation

$$\begin{pmatrix}\text{sech}\,x\\ \text{csch}\,x\end{pmatrix} F'' - 2\begin{pmatrix}\text{sech}\,x\tanh x\\ \text{csch}\,x\coth x\end{pmatrix} F' + \begin{pmatrix}(1+\alpha)\,\text{sech}\,x - 2\,\text{sech}^3 x\\ (1+\alpha)\,\text{csch}\,x + 2\,\text{csch}^3 x\end{pmatrix} F = 0, \tag{34a, b}$$

whose first two terms appear on the right-hand side of the identity

$$\left\{\begin{pmatrix}\text{sech}\,x\\ \text{csch}\,x\end{pmatrix} F\right\}'' = \begin{pmatrix}\text{sech}\,x\\ \text{csch}\,x\end{pmatrix} F'' - 2\begin{pmatrix}\text{sech}\,x\tanh x\\ \text{csch}\,x\coth x\end{pmatrix} F' + \begin{pmatrix}-\text{sech}^3 x + \text{sech}\,x\tanh^2 x\\ +\text{csch}^3 x + \text{csch}\,x\coth^2 x\end{pmatrix} F. \tag{35a, b}$$

Subtracting equations (34a, b) from equations (35a, b) one obtains, after some simplification,

$$\left\{\begin{pmatrix}\text{sech}\,x\\ \text{csch}\,x\end{pmatrix} F\right\}'' + \alpha\begin{pmatrix}\text{sech}\,x\\ \text{csch}\,x\end{pmatrix} F = 0, \tag{36a, b}$$

which is of the predicted form (33).

3.3. EFFECTIVE WAVENUMBER FOR UNIDIRECTIONAL PROPAGATION

The transformation of the wave equation for the acoustic velocity perturbation spectrum (32a, b) to the type (33) corresponds to the choice of variable

$$G(x) \equiv \begin{pmatrix}\text{sech}\,x\\ \text{csch}\,x\end{pmatrix} F(x) = \begin{matrix}\text{sech}\\ \text{csch}\end{matrix}\,(z/2l)\,V(z;\omega) = W(z;\omega), \tag{37a, b}$$

where $G(x)$ is the reduced velocity $W(z;\omega)$, defined by expression (7b) with the cross-section $S \sim \text{sech}\,x$, $\text{csch}\,x$ (28a, b), and $x = z/2l$ (31a); the latter change of variable proves the coincidence of expressions (33) and (21b). Note that the general method used to deduce expression (21a) applies to arbitrary horn shapes, and allows the prediction of which types have elementary exact solutions, and is clearer than the direct deduction ((31)–(37a, b)), which applies only to the inverse catenoidal ducts, and rests on the foreknowledge of their constant cut-off properties. The general equation (21a), or its verification for the inverse catenoidal ducts (37a, b), has the solutions:

$$W(z;\omega) = G(x) = \begin{cases} A\,e^{i\gamma x} + B\,e^{-i\gamma x} & \text{if } \alpha > 0 \quad (38a)\\ Ax + B & \text{if } \alpha = 0 \quad (38b)\\ A\cosh(\gamma x) + E\sinh(\gamma x) & \text{if } \alpha < 0 \quad (38c)\end{cases}$$

where γ denotes the square root of the modulus of expression (22b), viz. $\gamma \equiv |\alpha|^{1/2}$, and one has, in agreement with section 3.1, (a) propagating waves $\alpha > 0$ above the cut-off $\omega > \omega_v$, (c) non-oscillating modes $\alpha < 0$ below the cut-off $\omega < \omega_v$, and (b) transition fields $\alpha = 0$ at the cut-off $\omega = \omega_v$.

Considering frequencies above the cut-off $\omega > \omega_v$, for which propagation is possible, one has, from expression (22b),

$$\gamma \equiv \sqrt{\alpha} = 2lK, \qquad K \equiv (\omega/c)|1 - \omega_v^2/\omega^2|^{1/2}, \tag{39a, b}$$

where K is the effective wavenumber, since it (i) simplifies to the ordinary wavenumber $K \simeq \omega/c$ for frequencies much higher than the cut-off $\omega^2 \gg \omega_v^2$, (ii) is smaller than $\omega/c > K$ for intermediate frequencies $\omega > \omega_v$, and (iii) vanishes at the cut-off ($K = 0$ for $\omega = \omega_v$) when propagation becomes impossible. It is clear that the first/second terms of expression

(38a) correspond ($e^{\pm i\gamma x}=e^{\pm iKz}$) to a wave propagating in the direction of increasing/decreasing z, and, using expressions (38a) and (31a, b), one obtains the velocity perturbation spectrum

$$V_{\pm}(z;\omega)=V_0 \begin{matrix}\cosh\\ \sinh\end{matrix}(z/2l)\,e^{\pm iKz}, \tag{40a, b}$$

where (i) the upper/lower signs correspond to propagation in the directions of respectively increasing/decreasing z, (ii) the upper/lower coefficient corresponds respectively to the sech-horn/csch-baffle. The acoustic pressure perturbation spectrum,

$$P_{\pm}(z;\omega)=\rho c V_0\left\{\pm(Kc/\omega)\begin{matrix}\cosh\\ \sinh\end{matrix}(z/2l)+i(\omega_v/\omega)\begin{matrix}\sinh\\ \cosh\end{matrix}(z/2l)\right\}e^{\pm iKz}, \tag{41a, b}$$

follows from the substitution of expressions (28a, b) and (40a, b) into equation (6a).

From expressions (40a, b) and (41a, b) follows that the kinetic and compression energies (per unit volume),

$$E_v=\tfrac{1}{2}\rho|V(z;\omega)|^2=\tfrac{1}{2}\rho V_0^2\begin{matrix}\cosh^2\\ \sinh^2\end{matrix}(z/2l), \tag{42a}$$

$$E_p=\tfrac{1}{2}|P(z;\omega)|^2/\rho c^2=\tfrac{1}{2}\rho V_0^2\left\{\left(\frac{Kc}{\omega}\right)^2\begin{matrix}\cosh^2\\ \sinh^2\end{matrix}(z/2l)+\left(\frac{\omega_v}{\omega}\right)^2\begin{matrix}\sinh^2\\ \cosh^2\end{matrix}(z/2l)\right\}, \tag{42b}$$

are generally not equal, the reason being that the acoustic velocity and pressure satisfy distinct wave equation (5a, b), and thus evolve differently along the duct ((40a, b) and (41a, b)) violating the equipartition of energy; the latter only holds $E_v=E_p$ (42a, b) in the W.K.B.J. approximation, for frequencies much higher than the cut-off $(\omega_v/\omega)^2\to 0$, for which $(K_c/\omega)^2\to 1$ by expression (39b). Inserting the factor $e^{-i\omega t}$ (4a, b) in expressions (40a, b) and (41a, b) gives the exact acoustic velocity and pressure perturbations for acoustic waves of frequency ω in the sech-horn (upper coefficient)/csch-baffle (lower coefficient) as

$$V_{\pm}(z,t)=V_0\begin{matrix}\cosh\\ \sinh\end{matrix}(z/2l)\cos(Kz\pm\omega t), \tag{43a, b}$$

$$P_{\pm}(z,t)=\mp\rho c V_0\left\{\frac{Kc}{\omega}\begin{matrix}\cosh\\ \sinh\end{matrix}(z/2l)\cos(Kz\pm\omega t)-\frac{\omega_v}{\omega}\begin{matrix}\sinh\\ \cosh\end{matrix}(z/2l)\sin(Kz\pm\omega t\right\}, \tag{44a, b}$$

propagating in the direction of decreasing (upper sign)/increasing (lower sign) z.

4. WAVE FIELDS IN THE SECH-HORN AND CSCH-BAFFLE

The solution of the acoustic equations for the inverse catenoidal ducts (section 3.2) can be used to list the velocity and pressure perturbations in the cases of propagating waves above the cut-off (section 4.1), non-oscillating modes below the cut-off (section 4.3), and the transition fields between them at the cut-off frequency (section 4.2), given (a) the initial pressure and velocity for the sech-horn, or (b) the initial pressure and a suitable derivative of the velocity for the csch-baffle.

4.1. PROPAGATING WAVES ABOVE THE CUT-OFF FREQUENCY

The general solution for the velocity perturbation spectrum (31b) and (37b) of propagating waves (38a), above the cut-off frequency (39a, b), is a superposition of the waves propagating in opposite directions (40a),

$$V(z;\omega>\omega_v)=\cosh(z/2l)(A\,e^{iKz}+B\,e^{-iKz}), \tag{45a}$$

where A, B are constants of integration, and the sech-horn (28a) has been considered, the corresponding pressure perturbation spectrum (6a) being

$$P(z;\omega>\omega_v)=\rho c(cK/\omega)\cosh(z/2l)(A\,\mathrm{e}^{\mathrm{i}Kz}-B\,\mathrm{e}^{-\mathrm{i}Kz})$$
$$+\mathrm{i}\rho c(\omega_v/\omega)\sinh(z/2l)(A\,\mathrm{e}^{\mathrm{i}Kz}+B\,\mathrm{e}^{-\mathrm{i}Kz}). \tag{45b}$$

The constants of integration A, B, specify through A, $B\cosh(z/2l)$ the amplitudes of the velocity components propagating in the positive/negative directions, and are determined from two compatible initial, boundary or radiation conditions, e.g.: (i) in section 3.3 the radiation condition and initial velocity were used to set $A=V_0$ and $B=0$ in expressions (40a) and (41a), selecting propagation in one direction only, along an infinite duct; (ii) in general one has to consider the superposition of waves propagating in opposite directions (45a, b), with amplitudes determined from the velocity V, pressure P or impedance $Z\equiv P/VS$ at two sections $z=z_1$ and $z=z_2$, e.g., in the case of the finite duct $z_1\leqslant z\leqslant z_2$; (iii) as another example one can specify the initial velocity and pressure perturbation spectra,

$$V_0\equiv V(0;\omega>\omega_v)=A+B,\qquad P_0\equiv P(z;\omega>\omega_v)=\rho c(Kc/\omega)(A-B), \tag{46a, b}$$

showing that one has direct $A\neq 0$ and reflected $B\neq 0$ waves in an infinite duct if the initial disturbance is an arbitrary combination of velocity and pressure $P_0\neq\rho c(cK/\omega)V_0$. Thus one obtains, from expressions (45a, b) and (46a, b),

$$v(z,t)=\cosh(z/2a)\{V_0\cos Kz\cos\omega t+(P_0/\rho c)(\omega/Kc)\sin Kz\sin\omega t\}, \tag{47a}$$

$$p(z,t)=P_0\{\cosh(z/2l)\cos Kz-(\omega_v/Kc)\sinh(z/2l)\sin Kz\}\cos\omega t$$
$$+\rho cV_0\{(\omega_v/\omega)\sinh(z/2l)\cos Kz+(Kc/\omega)\cosh(z/2l)\sin Kz\}\sin\omega t, \tag{47b}$$

for the velocity and pressure perturbations of a wave of frequency $\omega>\omega_v=c/2l$ propagating in the sech-horn (28a), with the initial conditions (46a, b).

In the case of the csch-baffle (28b), one has the general propagating velocity and pressure perturbation spectra

$$V(z;\omega>\omega_v)=\sinh(z/2l)(A\,\mathrm{e}^{\mathrm{i}Kz}+B\,\mathrm{e}^{-\mathrm{i}Kz}), \tag{48a}$$

$$P(z;\omega>\omega_v)=\rho c(Kc/\omega)\sinh(z/2l)(A\,\mathrm{e}^{\mathrm{i}Kz}-B\,\mathrm{e}^{-\mathrm{i}Kz})$$
$$+\mathrm{i}\rho c(\omega_v/\omega)\cosh(z/2l)(A\,\mathrm{e}^{\mathrm{i}Kz}+B\,\mathrm{e}^{-\mathrm{i}Kz}). \tag{48b}$$

When specifying the initial conditions it should be noted that the velocity vanishes $V(0;\omega>\omega_v)=0$ because of the infinite flare $S(0)=\infty$ of the csch-baffle (28b) at $z=0$. Thus one can specify the initial pressure and the second derivative of velocity (or rate of dilatation $\mathrm{d}^2V/\mathrm{d}z^2$),

$$V_0''\equiv\{\mathrm{d}^2V(z;\omega>\omega_v)/\mathrm{d}z^2\}_{z=0}=\mathrm{i}K(A-B)l, \tag{49a}$$

$$P_0\equiv P(0;\omega>\omega_v)=\mathrm{i}\rho c(\omega_v/\omega)(A+B), \tag{49b}$$

to determine the constants A, B in expressions (48, b), and obtain

$$v(z,t)=\sinh(z/2l)\{(V_0''/K)\sin Kz\cos\omega t-(P_0/\rho c)(\omega/\omega_v)\cos Kz\sin\omega t\}, \tag{50a}$$

$$p(z,t)=P_0\{\cosh(z—2l)\cos Kz+(Kc/\omega_v)\sinh(z/2l)\sin Kz\}\cos\omega t$$
$$-\rho c^2(V_0''l/\omega)\{\sinh(z/2l)\cos Kz-(\omega_v/Kc)\cosh(z/2l)\sin Kz\}\sin\omega t, \tag{50b}$$

respectively, for the velocity and pressure perturbations of propagating waves of frequency $\omega>\omega_v=c/2l$ in the csch-baffle (28b) with initial conditions (49a, b).

4.2. TRANSITION FIELDS AT THE CUT-OFF CONDITION

At the cut-off frequency $\omega = \omega_v$ the effective wavenumber vanishes $K = 0$ (39b), and the two constants of integration A, B in expressions (45a) and (48a) reduce to one $A + B$, so that one must use the solution (38b) instead of (38a). For the sech-horn (28a) one obtains the velocity (31b) and (37a) and pressure (6a) perturbation spectra

$$V(z;\omega_v) = \cosh(z/2l)(Az/2l + B), \tag{51a}$$

$$P(z;\omega_v) = -i\rho c\{A\cosh(z/2l) - (B + Az/2l)\sinh(z/2l)\}, \tag{51b}$$

at the cut-off frequency $\omega_v = c/2l$. The constants of integration are determined from the initial velocity and pressure,

$$V_0 \equiv V(0;\omega_v) = B, \qquad P_0 \equiv P(0;\omega_v) = -i\rho c A, \tag{52a, b}$$

and lead to

$$v(z, t) = \cosh(z/2l)\{V_0\cos(ct/2l) + (P_0/\rho c)(z/2l)\sin(ct/2l)\}, \tag{53a}$$

$$p(z, t) = P_0\{\cosh(z/2l) - (z/2l)\sinh(z/2l)\}\cos(ct/2l) - \rho c V_0 \sinh(z/2l)\sin(ct/2l), \tag{53b}$$

respectively, for the velocity and pressure perturbations of transition waves at the cut-off frequency $\omega = \omega_v = c/2l$ in the sech-horn (28a) with initial conditions (51a, b).

In the case of the csch-baffle (28b) the transition velocity and pressure perturbation spectra are given by

$$V(z;\omega_v) = \sinh(z/2l)\{A(z/2l) + B\}, \tag{54a}$$

$$P(z;\omega_v) = -i\rho c\{A\sinh(z/2l) - (Az/2l + B)\cosh(z/2l)\}, \tag{54b}$$

at the cut-off frequency $\omega_v = c/2l$. When determining the constants A, B from the initial conditions it should be borne in mind that, as a consequence of the infinite flare $S(0) = \infty$ of the csch-baffle at $z = 0$ (i) the initial velocity is zero $V(0;\omega_v) = 0$, and determines nothing, (ii) the first derivative of the velocity (or dilatation) $V_0' \equiv \{\mathrm{d}V(z;\omega_v)/\mathrm{d}z\}_{z=0}$ is redundant with the initial pressure $P_0 \equiv P(0;\omega_v)$, as can be seen from $P_0 = -i(\rho c^2/\omega)V_0'$ in expression (6a), or from the fact that both would determine the constant B through $P_0 = i\rho c B$ and $V_0' = B/2l$, so that they cannot be chosen independently, and (iii) the second derivative of the velocity or rate of dilatation V_0'' can be chosen independently of the pressure P_0, since it specifies the "other" constant of integration A through

$$V_0'' \equiv \{\mathrm{d}^2V(z;\omega_v)/\mathrm{d}z^2\}_{z=0} = A/2l^2, \qquad P_0 \equiv P(0;\omega_v) = i\rho c B. \tag{55a, b}$$

Substituting for A, B from expressions (55a, b) in expressions (54a, b) yields:

$$v(z, t) = \sinh(z/2l)\{lV_0''z\cos(ct/2l) - (P_0/\rho c)\sin(ct/2l)\}, \tag{56a}$$

$$p(z, t) = P_0\cosh(z/2l)\cos(ct/2l) - 2\rho c l^2 V_0''\{\sinh(z/2l) - (z/2l)\cosh(z/2l)\}\sin(ct/2l), \tag{56b}$$

respectively, for velocity and pressure perturbations of transition waves at the cut-off frequency $\omega_v = c/2l$ in the csch-baffle (28b) with initial conditions (55a, b).

4.3. NON-OSCILLATING MODES BELOW THE CUT-OFF

Below the cut-off one should use the solution (38c), where $\gamma \equiv \sqrt{|\alpha|}$,

$$\gamma \equiv |1 - (\omega/\omega_v)^2|^{1/2} = |1 - 4\omega^2 l^2/c^2|^{1/2}, \tag{57a, b}$$

where α is given by equation (22b) both for the sech-horn (28a) and csch-baffle (28b).

The velocity (31a, b) and (37a, b) and pressure (6a) perturbation spectra are given by

$$V(z;\omega<\omega_v)=\genfrac{}{}{0pt}{}{\cosh}{\sinh}(z/2l)\left\{A\genfrac{}{}{0pt}{}{\cosh}{\cosh}(\gamma z/2l)+B\genfrac{}{}{0pt}{}{\sinh}{\sinh}(\gamma z/2l)\right\}, \tag{58a}$$

$$\begin{aligned}P(z;\omega<\omega_v)=-\mathrm{i}\rho c(\omega_v/\omega)\left\{\genfrac{}{}{0pt}{}{\cosh}{\sinh}(z/2l)\left\{A\genfrac{}{}{0pt}{}{\sinh}{\sinh}(\gamma z/2l)+B\genfrac{}{}{0pt}{}{\cosh}{\cosh}(\gamma z/2l)\right\}\right.\\ \left.-\genfrac{}{}{0pt}{}{\sinh}{\cosh}(z/2l)\left\{A\genfrac{}{}{0pt}{}{\cosh}{\cosh}(\gamma z/2l)+B\genfrac{}{}{0pt}{}{\sinh}{\sinh}(\gamma z/2l)\right\}\right\},\end{aligned} \tag{58b}$$

below the cut-off. The wave fields simplify at the cut-off because $\gamma=0$ at $\omega=\omega_v$ (57a, b), although in this case only one constant remains, so that the solution must be replaced by that given in section 4.2. The wave fields (58a, b) below the cut-off are *not* evanescent, since the exponential decrease in cross-section $S(z)$ for $z\to\infty$ in (28a, b) causes the wave amplitude to grow, having the factor $S^{-1/2}$ of propagating waves (45) and (48a, b) multiplied by another smaller factor cosh, $\sinh(\gamma x)<\cosh,\sinh(x)$ since $\gamma<1$. The distinction is that the wave fields below the cut-off (58a, b) are non-oscillating in space, whereas above the cut-off (45a, b) and (48a, b) there are nodes at half-wavelength interval $\Lambda\equiv\lambda/2=\pi/K$. The spacing between the nodes tends to infinity $\Lambda\to\infty$ as the cut-off condition is approached, $\omega\to\omega_v$, and propagation becomes impossible, $K\to 0$.

In the case of the sech-horn, which corresponds to the upper coefficients in equation (58a, b), the constants A, B are determined from the initial velocity and pressure perturbation spectra (as in expressions (45a, b)):

$$V_0\equiv V(0;\omega<\omega_v)=A,\qquad P_0\equiv P(0;\omega<\omega_v)=-\mathrm{i}\rho c(\omega_v/\omega)\gamma B, \tag{59a, b}$$

leading to the expressions,

$$\begin{aligned}v(z,t)=\cosh(z/2l)\{&V_0\cosh(\gamma z/2l)\cos(\omega t)\\&+\gamma^{-1}(P_0/\rho c)(\omega/\omega_0)\sinh(\gamma z/2l)\sin(\omega t)\},\end{aligned} \tag{60a}$$

$$\begin{aligned}p(z,t)=P_0\{&\cosh(z/2l)\cosh(\gamma z/2l)-\gamma^{-1}\sinh(z/2l)\sinh(\gamma z/2l)\}\cos(\omega t)\\&+\rho cV_0(\omega_v/\omega)\{\gamma\cosh(z/2l)\sinh(\gamma z/2l)\\&-\sinh(z/2l)\cosh(\gamma z/2l)\}\sin(\omega t),\end{aligned} \tag{60b}$$

for the velocity and pressure perturbations of non-oscillating waves of frequency $\omega<\omega_v=c/2l$ in the sech-horn (28a) with initial conditions (59a, b).

In the case of the csch-baffle (28b) the infinite flare $S(0)=\infty$ at $z=0$ implies (as in section 4.2) that the velocity vanishes, the first derivative or dilatation is determined by the pressure, and thus only the second derivative of the velocity, or rate of dilatation, can be specified independently of the pressure (as in expressions (55a, b)),

$$V_0''\equiv\{\mathrm{d}^2V(z;\omega<\omega_v)/\mathrm{d}z^2\}_{z=0}=\gamma B/2l^2,\qquad P_0\equiv P(0;\omega<\omega_v)=\mathrm{i}\rho c(\omega_v/\omega)A, \tag{61a, b}$$

where expressions (57a, b) have been used, with the lower coefficients. Thus one obtains

$$\begin{aligned}v(z,t)=\{&(2lV_0''/\gamma)\sinh(\gamma z/2l)\cos(\omega t)\\&-(P_0/\rho c)(\omega/\omega_v)\cosh(\gamma z/2l)\sin(\omega t)\}\sinh(z/2l),\end{aligned} \tag{62a}$$

$$\begin{aligned}p(z,t)=P_0\{&\cosh(z/2l)\cosh(\gamma z/2l)-\gamma\sinh(z/2l)\sinh(\gamma z/2l)\}\cos(\omega t)\\&-2l^2V_0''\rho c(\omega_v/\omega)\{\sinh(z/2l)\cosh(\gamma z/2l)\\&-\gamma^{-1}\cosh(z/2l)\sinh(\gamma z/2l)\}\sin(\omega t),\end{aligned} \tag{62b}$$

for the velocity and pressure perturbations of non-oscillating modes of frequency $\omega < \omega_v = c/2l$ in the csch-baffle (28b) with initial conditions (61a, b).

5. DISCUSSION

Dimensionless representations of the frequency and wavenumber (section 5.1) and of the geometrical and acoustic properties of ducts (section 5.2) are introduced, before interpreting (section 5.3) the plots of the sound fields in the sech-horn (Figure 1(a)) and in the csch-baffle (Figure 1(b)).

5.1. IMPEDANCE AND DIMENSIONLESS FREQUENCY AND WAVENUMBER

From the point of view of the design of acoustic devices, viz., horn engineering [41–50], the assessment of the performance of the sech-horn and csch-baffle can be performed by methods similar to those described extensively in the literature about other shapes [22–27, 30–37], taking as starting point the impedance, which is given, from expressions (40a, b) and (41a, b), as

$$Z_{\pm}(z;\omega) \equiv P_{\pm}(z;\omega > \omega_v)/\{S(z)V_{\pm}(z;\omega > \omega_v)\}$$
$$= (\rho c/S_0)\left\{\pm(Kc/\omega)\frac{\cosh^2}{\sinh^2}(z/2l) - (\mathrm{i}/2)(\omega_v/\omega)\sinh(z/l)\right\}, \tag{63}$$

where (i) the upper/lower sign corresponds to propagation in the positive/negative z-direction, respectively, (ii) the upper-lower coefficients apply, respectively, to the sech-horn (28a)/csch-baffle (28b). A detailed discussion of the potential uses of these horn and baffle shapes is beyond the scope of the present work, and therefore only a brief illustration, in Figures 1(a) and (b), of the physical properties of sound in ducts of non-uniform cross-section, in connexion with these two shapes and a variety of frequencies, is given here.

The acoustics of the sech-horn and csch-baffle is specified by a single dimensionless parameter, namely, the ratio of the wave frequency ω to the cut-off frequency ω_v:

$$\Delta \equiv \omega l/c = 2\pi l/\lambda = \omega/2\omega_v = 0{\cdot}5,\ 1{\cdot}0,\ 2{\cdot}0,\ \infty, \tag{64a}$$

where the factor 1/2 has been used so that Δ can be identified with the compactness parameter $\Delta \equiv kl$, where $k \equiv \omega/c$ is the ordinary wavenumber, as used commonly in wave scattering theory, to compare the wavelength λ to lengthscale l of the duct. In the W.K.B.J. approximation, or ray limit, the cross-section varies slowly on the scale of a wavelength $\lambda^2 \ll l^2$, and the parameter (64a) is large, $\Delta^2 \gg 1$; the opposite limit is the cut-off condition $\omega = \omega_v$, when the non-uniform cross-section renders propagation impossible, and the parameter takes the minimum value $\Delta = 1/2$. The two other, intermediate values $\Delta = 1, 2$, given in expression (64a), of the compactness parameter, or dimensionless frequency, correspond to propagating waves for which the effects of variation of the cross-section of the ducts are considerable.

Another, related dimensionless quantity is the ratio of the effective wavenumber K defined in expression (39b) to the ordinary wavenumber $k \equiv \omega/c$:

$$\delta \equiv Kc/\omega = |1 - \omega_v^2/\omega^2|^{1/2} = |1 - 1/(4\Delta^2)|^{1/2} = 0{\cdot}000,\ 0{\cdot}866,\ 0{\cdot}968,\ 1{\cdot}000; \tag{64b}$$

the effective wavenumber K embodies the effects of the cut-off frequency associated with the non-uniformity of the cross-section of the duct, unlike the ordinary wavenumber k, which refers to plane waves in free space or in a uniform tube. It would be possible to define a propagating compactness Kl, which unlike expression (64a) vanishes at the

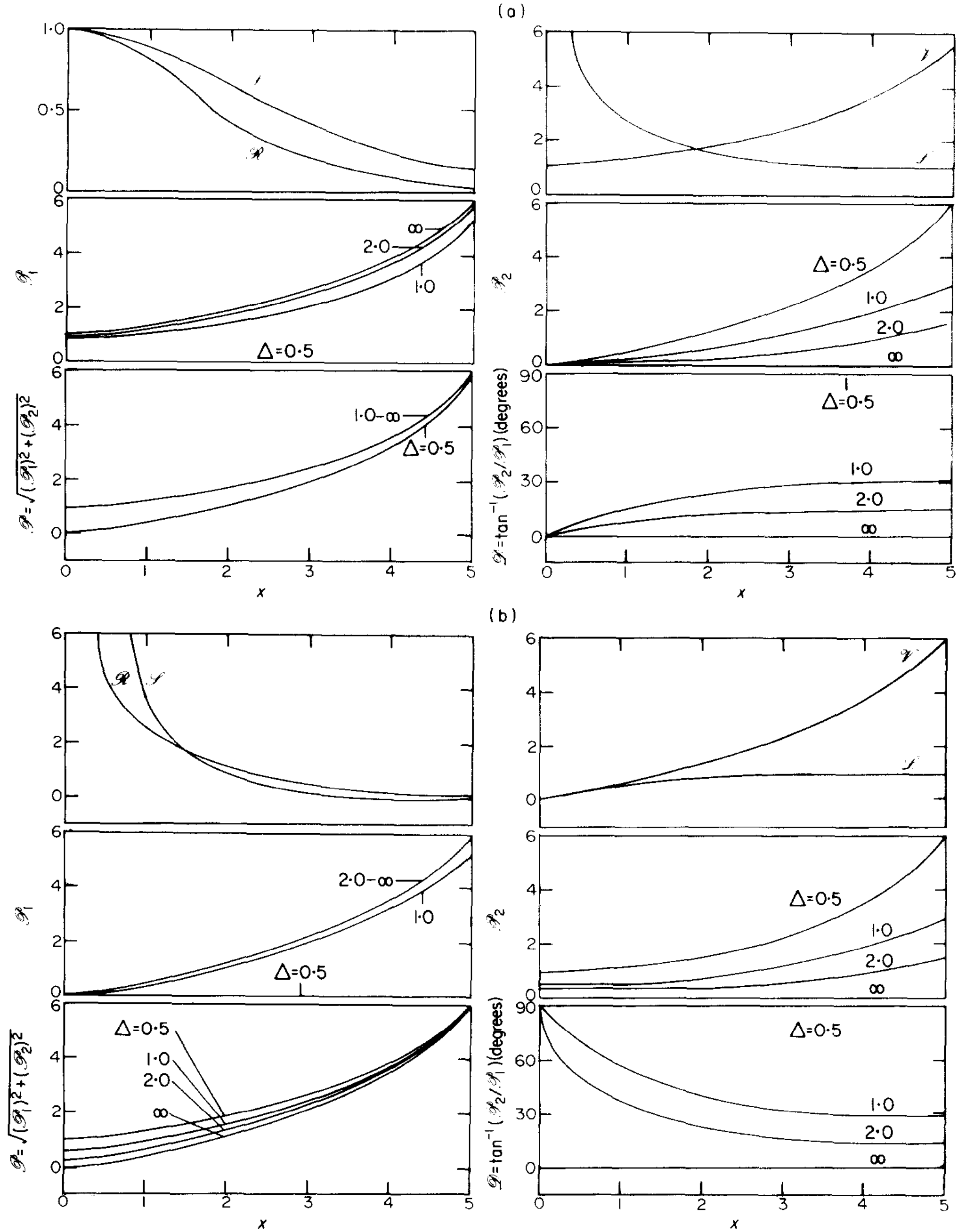

Figure 1. Propagation of the fundamental, longitudinal acoustic mode in the (a) sech-horn and (b) csch-baffle, described by plotting against dimensionless axial coordinate $x \equiv z/2l$ acoustic and geometric variables, rendered dimensionless by dividing either by asymptotic values (subscript ∞) or by initial values (subscript 0), the latter at station $z = 0$ for the sech-horn, and $x_0 = z_0/l = 1{\cdot}76$ for the csch-baffles. Top left, cross-section $\mathcal{S} \equiv S/S_0$ and radius $\mathcal{R} \equiv \sqrt{\mathcal{S}}$ (axisymmetry assumed); top right, length scale $\mathcal{L} \equiv L/L_\infty$ for cross-sectional variation $L \equiv S/S'$ and ratio of amplitude of acoustic velocity at station z to initial value $\mathcal{V} \equiv |V/V_0|$; centre left, primary pressure field $\mathcal{P}_1 \equiv \mathrm{Re}\,(P/\rho c V_0)$, in phase with velocity; centre right, secondary pressure field $\mathcal{P}_2 \equiv \mathrm{Im}\,(P/\rho c V_0)$, out of phase with velocity; bottom left, total pressure field $\mathcal{P} \equiv \sqrt{(\mathcal{P}_1)^2 + (\mathcal{P}_2)^2}$, at station z divided by initial value for plane wave; bottom right, phase difference $\mathcal{D} \equiv \tan^{-1}(\mathcal{P}_2/\mathcal{P}_1)$ between pressure and velocity. The first two plots (top rows) are independent of frequency, and the last four (centre and bottom rows) include curves for four values of the dimensionless frequency $\Delta = 0{\cdot}5$, $1{\cdot}0$, $2{\cdot}0$, ∞, corresponding, respectively, to the cut-off frequency $\omega = \omega_v$, twice and four times this value $\omega = 2\omega_v$, $4\omega_v$, and the ray limit $\omega \to \infty$.

cut-off, and is related to Δ (64a) and δ (64b) by $Kl = \Delta\delta$. The advantage of the dimensionless wavenumber (64b) when compared with the dimensionless frequency (64a) for propagating waves, is that the former has a much smaller range of variation $0 < \delta < 1$ (than the latter $0{\cdot}5 < \Delta < \infty$), with the cut-off condition corresponding to zero and the W.K.B.J. approximation or ray limit to unity, the latter value being approached quickly for frequencies away from the cut-off.

5.2. GEOMETRICAL AND ACOUSTIC PROPERTIES OF DUCTS

The properties of the sech-horn and csch-baffle are illustrated, in Figures 1(a) and (b), in terms of a number of dimensionless quantities, of both geometrical and acoustic nature, which are plotted as functions of the dimensionless longitudinal co-ordinate $x \equiv z/2l$ (31a). The cross-section,

$$\mathcal{S} \equiv S(z)/S_0 = \operatorname{sech}^2(z/2l), \operatorname{csch}^2(z/2l), \tag{65a, b}$$

or, for an axisymmetric duct shape, the radius,

$$\mathcal{R} \equiv \mathcal{S}^{1/2} = \operatorname{sech}(z/2l), \operatorname{csch}(z/2l), \tag{66a, b}$$

are made dimensionless by dividing by the values at the station $z = 0$ for the sech-horn and $z_0 = 1{\cdot}76l$ for the csch-baffle. The length scale (10a) is negative because the ducts are convergent, and is plotted with reversed sign,

$$\mathcal{L} \equiv -L(z)/l = \coth(z/2l), \tanh(z/2l), \tag{67a, b}$$

and is normalized to the length parameter l.

The amplitude of the velocity perturbation spectrum (40a, b), for a wave of frequency ω at the station z, is divided by the reference value $V_0(\omega)$:

$$\mathcal{V} \equiv |V_\pm(z;\omega > \omega_v)/V_0(\omega)| = \cosh(z/2l), \sinh(z/2l), \tag{68a, b}$$

with V_0 calculated at $z = 0$ for the sech-horn $V_0(\omega) \equiv V_\pm(0;\omega > \omega_v)$, and at the station $z_0/l = 1{\cdot}76$ for the sech-baffle $V_0(\omega) \equiv V_\pm(z_0;\omega > \omega_v)$. The pressure field (41a, b) may be rendered dimensionless dividing by $\rho c V_0(\omega)$, and since it is generally out-of-phase with the velocity, it is decomposed into two parts,

$$\mathcal{P}_1 \equiv \operatorname{Re}\{\mp P_\pm(z;\omega > \omega_v)/\rho c V_0(\omega)\} = \delta\{\cosh(z/2l), \sinh(z/2l)\}, \tag{69a}$$

$$\mathcal{P}_2 \equiv \operatorname{Im}\{P_\pm(z;\omega > \omega_v)/\rho c V_0(\omega)\} = (2\Delta)^{-1}\{\sinh z/2l), \cosh(z/2l)\}, \tag{69b}$$

$\mathcal{P}_1$ being in-phase with the reference velocity $V_0(\omega)$, and $\mathcal{P}_2$ out-of-phase by $\pi/2$. In both cases the upper/lower sign corresponds (45a, b) to propagation in the positive/negative z (or x-) direction, respectively.

The quantities (69a, b) specify the primary and secondary acoustic fields, corresponding, through $P = SVZ$, to the real and imaginary parts of the impedance $Z \equiv Z_1 + iZ_2$, i.e., the resistance Z_1 and inductance Z_2, of which $\mathcal{P}_1$ and $\mathcal{P}_2$ are the dimensionless counterparts. The primary and secondary acoustic fields add up to the total dimensionless pressure amplitude,

$$\mathcal{P} \equiv |P_\pm(z;\omega > \omega_v)/\rho c V_0(\omega)| = \{(\mathcal{P}_1)^2 + (\mathcal{P}_2)^2\}^{1/2}, \tag{70a, b}$$

which coincides with the acoustic pressure perturbation spectrum for the non-uniform duct divided by its value for a plane wave in a tube of constant cross-section S_0. The relative importance of the primary $\mathscr{P}_1$ and secondary $\mathscr{P}_2$ acoustic fields measures, through

$$\mathscr{D} \equiv \arg\{P_\pm(z;\omega>\omega_c) - \arg V_\pm(z;\omega>\omega_c)\}$$
$$= \tan^{-1}[(2\Delta\delta)^{-1}\{\tanh(z/2l), \coth(z/2l)\}], \qquad (71a, b)$$

the phase difference between the pressure and velocity.

5.3. PRIMARY AND SECONDARY ACOUSTIC FIELDS

The sets of six plots each in Figure 1(a) for the sech-horn and in Figure 1(b) for the csch-baffle, respectively, may be interpreted from left to right and top to bottom. In both cases (top left) the cross-section $\mathscr{S}$ (65a, b) and radius $\mathscr{R}$ (66a, b) show that the ducts converge exponentially on a length scale l for large distance $z^2 \gg l^2$, but for small coordinate $z^2 \ll l^2$ the sech-horn flares down smoothly to a finite area at $z=0$, whereas the csch-baffle flares up to an infinite area, and the transition to its mirror image is not smooth. The length scale $\mathscr{L}$ (67a, b) tends to a constant $-l$ for large distance $z^2 \gg l^2$, and for smaller z increases due to the flare down of the sech-horn, to ∞ at $z=0$ where the cross-section is constant to order z^2/l^2, and decreases due to the flare up of the csch-baffle, to 0 as $z \to 0$ (top right). The amplitude of the velocity $\mathscr{V}$ ((68a, b); also top right) diverges for large z as the tube converges, and reduces for small z to a finite value for the sech-horn (of finite area) at $z=0$, and to zero at $z=0$ for the csch-baffle (of infinite area).

The primary pressure field $\mathscr{P}_1$ ((69a); center left) is uniformly zero at the cut-off, when the pressure is 90° out of phase to the velocity, and for propagating frequencies away from the cut-off it takes values close to the W.K.B.J. limit, increasing as the duct converges, from an initial value which is finite for the sech-horn and zero for the csch-baffle: i.e., evolves in a pattern similar to the velocity's. The secondary pressure field $\mathscr{P}_2$ ((69b); center right) is zero in the ray limit, when the pressure is in phase with the velocity, and generally increases strongly as the frequency takes values near to the cut-off; it increases as the duct converges, from an initial value which is zero for the sech-horn, since the finite cross-section implies that pressure and velocity are initially in phase, and non-zero for the csch-baffle, since due to the infinite flare, the pressure and velocity start out of phase. The preceding differences in the behaviour of the primary and secondary pressure fields with regard to the frequency, with the former predominating towards the ray limit (propagating plane wave) and the latter towards the cut-off condition (non-oscillating modes), implies that the total pressure $\mathscr{P}$ ((70a, b); bottom left) is not too sensitive to frequency; it increases as the duct converges, and is asymptotically independent of frequency, but initially it is larger for higher frequencies for the sech-horn, and larger for lower frequencies for the csch-baffle, since in the former/latter respectively for primary/secondary field predominates initially.

The contrasts between the primary and secondary pressure affect the phase difference $\mathscr{D}$ (71a, b) between pressure and velocity (bottom right) since (i) in both cases the phase difference vanishes in the W.K.B.J. approximation or ray limit equivalent to absence of reflections from the walls, and it takes the constant, uniform value 90° at the cut-off when propagation becomes impossible, (ii) for intermediate frequencies, the phase always starts at 0° for the sech-horn and at 90° for the csch-baffle, since for the former the finite, smooth initial cross-section implies that the velocity and pressure start in phase, whereas for the latter the infinite area implies that they start 90° out-of-phase, as modes in a cavity, (iii) the phase tends rapidly to (i.e., approaches in about three length scales $z/l \geqslant 3$) the asymptotic value $\theta_\infty \equiv \mathscr{D}(\infty) = \tan^{-1}(2\delta\Delta)$, which is the same both for the sech-horn and csch-baffle, and depends only on the ratio of wave to cut-off frequency, and (iv) the values $\theta_\infty = 30{\cdot}0°$, $14{\cdot}5°$, respectively, for frequencies twice and four times the cut-off frequency,

$\omega = 2\omega_v, 4\omega_v$, show that the asymptotic phase difference is larger the closer the frequency is to the cut-off, and that, moderately away from the cut-off, it approaches the W.K.B.J. or ray or in phase limit.

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APPENDIX: LIST OF SYMBOLS

c	sound speed
k	ordinary wavenumber, $k \equiv \omega / c$
l	constant length parameter for ducts with constant cut-off, (14) and (18a, b)
$p(z, t)$	acoustic pressure perturbation
t	time
$v(z, t)$	acoustic velocity perturbation
x	dimensionless co-ordinate, $x \equiv z / l$
z	co-ordinate along the axis of the horn
z_0	station $2l \sinh^{-1}(1) = 1{\cdot}76l$
A, B	constants of integration
$F(x)$	velocity perturbation spectrum $V(z; \omega)$ in terms of x
$G(x)$	reduced velocity $W(z; \omega)$ in terms of x
J_v, J_p	wave invariants for sound velocity and pressure, (8) and (9a, b)
K	effective wavenumber, (39b)
$L(z)$	length scale for cross-sectional variation $L \equiv S/S'$
$P(z; \omega)$	acoustic pressure perturbation spectrum, (4a)
P_0	initial pressure perturbation spectrum $P(0; \omega)$
$Q(z; \omega)$	reduced acoustic pressure spectrum $\sqrt{S(z)}P(z; \omega)$, (7a)

$S(z)$	cross-sectional area at station z
S_0	initial cross-sectional area, or cross-sectional area at station z_0
$V(z;\omega)$	acoustic velocity perturbation spectrum, (4b)
V_0	initial velocity perturbation spectrum $V(0;\omega)$
V_0', V_0''	initial value of first, second derivative of velocity perturbation spectrum $\mathrm{d}^n V(z;\omega)/\mathrm{d}z^n\|_{z=0}$, with $n=1,2$
$W(z;\omega)$	reduced acoustic velocity spectrum $\sqrt{S(z)}V(z;\omega)$, (7b)
$Z(z;\omega)$	impedance $P(z;\omega)/S(z)V(z,\omega)$
α	dimensionless wave invariant for catenoidal and inverse ducts (22a, b)
β	dimensionless wave invariant for sinusoidal and inverse ducts (27a, b)
γ	parameter for standing modes
δ	dimensionless wavenumber, $\delta \equiv K/k$, (64b)
ρ	mean mass density
ω	wave frequency
ω_p, ω_v	cut-off frequency for acoustic pressure, velocity
Δ	dimensionless frequency, $\Delta \equiv \omega/2\omega_v$, (64a)
$\mathscr{D}$	phase difference between acoustic pressure and velocity, (71a, b)
$\mathscr{L}$	minus the dimensionless length scale of cross-sectional variation, (67a, b)
$\mathscr{P}$	dimensionless amplitude of acoustic pressure, (70a, b)
$\mathscr{P}_1, \mathscr{P}_2$	dimensionless pressure component in-phase/out-of-phase with velocity, (69a, b)
$\mathscr{R}$	dimensionless radius for axisymmetric ducts, (66a, b)
$\mathscr{S}$	dimensionless cross-section, (65a, b)
$\mathscr{V}$	dimensionless amplitude of acoustic velocity, (68a, b)