ACTA ACUSTICA UNITED WITH ACUSTICA
Vol. 97 (2011) 699 – 707

DOI 10.3813/AAA.918449

The Application of the Dual Surface Method to Treat the Nonuniqueness in Solving Acoustic Exterior Problems

A. Mohsen[1], R. Piscoya[2], M. Ochmann[2]

[1] Eng. Math. & Phys. Dept., Eng. Faculty, Cairo University, Giza 12211, Egypt

[2] Beuth Hochschule für Technik, University of Applied Sciences, FB II, Research Group Computational Acoustics, Berlin 13353, Germany. piscoya@beuth-hochschule.de

Summary

The problem of nonuniqueness (NU) of the solution of exterior acoustic problems via boundary integral equations (BIEs) is studied. The application of the dual surface method, used in electromagnetic problems, to exterior acoustic problems is studied. The dual surface integral equations, although identical in form and comparable in complexity to the original surface integral equations, provide a unique solution at all real frequencies. The conditions and the proof of uniqueness are outlined. Applications of the method are given for the scattering as well as the radiation from three different structures. We consider normalized frequencies up to $ka \sim 22$, where "a" is a typical dimension of the structure.

PACS no. 43.20.Fn, 43.40.Rj

1. Introduction

Surface integral equation (SIE) treatment of exterior acoustic problems reduces the dimension of the problem by one and provides a direct implementation of the radiation and boundary conditions. However, the solution of the SIEs is not unique at internal resonances [1]. Methods to modify or reformulate the solution procedures to insure uniqueness over a range of wavenumbers or at all wavenumbers have been a topic of theoretical and practical interest [2, 3, 4]. Some methods involve relatively more operators which may take considerably more programming and computer time or require a special procedure to handle hypersingular integrals or to properly select the additional equations.

In this paper an application of the Dual Surface (DS) method [5, 6, 7, 8, 9] to acoustic radiation and scattering problems is considered. Although the method has been applied for quite some time in electromagnetics, it is not cited in later computational acoustic publications e.g. [2, 3, 4]. The method is here reintroduced in relation to known conventional methods and then applied to some acoustic radiation and scattering problems. The Dual Surface Integral Equations (DSIEs) have the same form and are comparable in complexity to the original SIE, it ensures a unique solution at all real frequencies. The conditions and the proof of uniqueness, which are based on the analysis of electromagnetic problems, are outlined.

Received 8 October 2010,
accepted 10 March 2011.

Tests are given for plane wave scattering by hard structures. Accurate results are obtained for normalized frequencies up to $ka \sim 20$. For scattering by a soft sphere, accurate results are also obtained using Fredholm integral equations of the first and second kinds at frequencies near internal resonances. Also the radiation from the vibrating structures is considered and accurate results are obtained.

2. Integral representations of the solution

Let V_i denote a bounded domain in three dimensions with a boundary Σ which is a closed surface. We denote the exterior of Σ by V_O. A suppressed time variation in the form $\exp(-\mathrm{i}\omega t)$ is assumed. It is convenient to introduce the following notations:

$$S\{\phi\} \equiv \int_\Sigma \phi(q)\, G(p,q)\, \mathrm{d}s_q, \tag{1a}$$

$$D\{\phi\} \equiv \int_\Sigma \phi(q)\, \partial_{nq} G(p,q)\, \mathrm{d}s_q, \tag{1b}$$

$$K\{\phi\} \equiv \partial_{nq} S\{\phi\}, \tag{1c}$$

$$M\{\phi\} \equiv \partial_{nq} D\{\phi\}, \tag{1d}$$

where ∂_n denotes outward normal derivative along $\boldsymbol{n}$. Here $G = \exp(\mathrm{i}kR)/(4\pi R)$ is the free space Green's function, $R = |\boldsymbol{p} - \boldsymbol{q}|$ and $\boldsymbol{p}$ and $\boldsymbol{q}$ denote a field and a surface point, respectively. $S\{.\}$ and $D\{.\}$ are the single and double layer operators, respectively. We note that equation (1d) involves hypersingular integrands ($\partial_{nq} D\{\phi\} \sim \int (1/R^3)\, \mathrm{d}s_q$).

Let U denote the scalar potential, then applying Green's second identity, we get the Helmholtz integral Formula (HIF) [10]:

$$D\{u\} - S\{u\} = \begin{cases} U & P \in V_O, \\ c(p)u & P \in \Sigma \\ 0 & P \in V_i, \end{cases} \tag{2}$$

where u is the surface value of U, $v = \partial_n u$ and $c(p)$ is given by

$$c(p) = 1 + \int_\Sigma \partial_{np}(1/R)\,\mathrm{d}s_q/(4\pi). \tag{2a}$$

Equation (2a) implies that the surface Σ may have a non-smooth geometry at edges and corners. At smooth points, $c(p) = 0.5$.

Upon invoking the appropriate boundary condition, we are led to an integral equation in the surface wave potential or its normal derivative. For the case of scattering of a potential field U^i incident on a smooth Σ, we may write

$$U = U^i + D\{u\} - S\{v\}. \tag{3}$$

In the limit, this equation and its normal derivative yield on Σ:

$$u/2 = u^i + D\{u\} - S\{v\}, \tag{4}$$

$$v/2 = v^i + M\{u\} - K\{v\}. \tag{5}$$

Thus for soft scattering ($u = 0$) we have

$$v/2 + K\{v\} = v^i, \tag{6a}$$

$$S\{v\} = u^i, \tag{6b}$$

while for hard scattering ($v = 0$), we have

$$u/2 - D\{u\} = u^i. \tag{7}$$

The Interior Helmholtz Field Equations (IHFEs) are given by

$$U^i + \underline{D}\{u\} - \underline{S}\{v\} = 0, \tag{8}$$

where in $\underline{D}\{u\}$ and $\underline{S}\{v\}$ the field points lie in the interior.

3. The nonuniqueness problem

While the original boundary value problem has a unique solution, the corresponding BIE may not be uniquely solvable at certain values of the wavenumber corresponding to the adjoint interior problem. This gives rise to analytical complications and considerable difficulty in the numerical treatment of the problem. While the integral equation fails only at a discrete point set of wavenumbers, the approximating linear equations become ill-conditioned when k is merely in the vicinity of a critical value. This is evident in the problem of a hard sphere, where the nonuniqueness effect is clearly evident at $ka = 22.602$ which is not at, but close to, the resonant frequencies. Under these conditions, severe loss of accuracy will be experienced. As the wavenumber k increases, so also does the density of critical values, and hence it becomes increasingly difficult to get an accurate solution.

The uniqueness of the solutions to the above equations was thoroughly discussed by Burton [1]. Thus the solution of (6a) is not unique if $k \in \{k_N\}$, where $\{k_N\}$ is the set of eigenvalues for the interior Neumann problem. At these wavenumbers the homogeneous equation adjoint to (6a) has nontrivial solutions. On the other hand, the solutions of (6b) and (7) are not unique when $k \in \{k_D\}$, where $\{k_D\}$ is the set of eigenvalues of the interior Dirichlet problem.

The NU problem may be detected via calculating the pivot ratio in Gauss elimination [11], monitoring the condition number of the resulting matrix [12], evaluating the minimum SVD [13] or testing the level of interior fields [10].

Several approaches have been devised for surmounting these defects and they are discussed in [2, 3, 4], [14] with references to previous contributions. These methods include the Burton and Miller (B&M) (composite, combined, Helmholtz Gradient) field formulation, the combined source (mixed potential, modified Green's function) method, the use of interior Helmholtz integral relations and the source simulation (wave superposition, method of fundamental solutions) technique. Some methods involve more operators which may take considerably more programming and computer time or require a special procedure to handle hypersingular integrals or select the proper additional equations.

In the present work, the application of the Dual Surface (DS) method [5, 6, 7, 8, 9], used in electromagnetic problems, to exterior acoustic problems is considered. The Dual Surface Integral Equations (DSIEs), ensure the uniqueness of the solution while maintaining the simplicity of the original SIE. The conditions and the proof of uniqueness follow from the electromagnetic case and are outlined. The method is presented next in relation to the combined Helmholtz integral equation formulation (CHIEF) [15] and the Burton and Miller (B&M) method [16].

4. The CHIEF, Burton-Miller and the dual surface methods

The two main conventional techniques that have been applied to exterior acoustic problems in order to avoid the NU problem are CHIEF and B&M methods.

In CHIEF, the SIEs (equations 4 or 5) are augmented by imposing some IHFEs (equation 8) at properly selected interior points. The uniqueness requires that the interior points (CHIEF points) do not lie on internal nodal surfaces. The resulting overdetermined system can then be solved using a least squares [15] or Lagrange multipliers [17] techniques. Various modifications and improvements have been proposed e.g. [18, 19, 20] mainly to ensure uniqueness in spite of the restriction imposed on interior points.

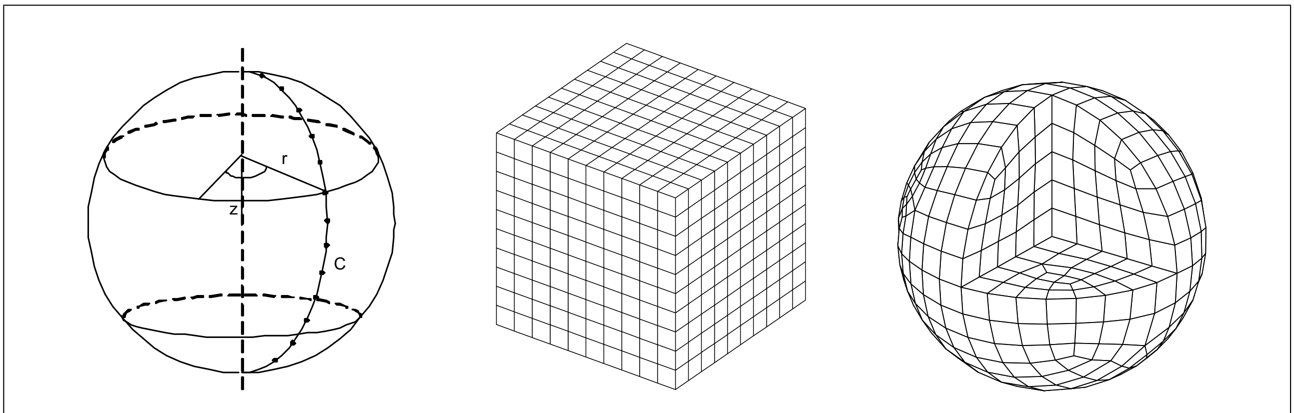

Figure 1. Finite element models of the test objects. Left: axisymmetric sphere; middle: cube; right: cat's eye.

In the B&M method, one uses a complex linear combination of the SIEs (4) and (5)

$$u/2 - D\{u\} - \gamma M\{u\} \\ +S\{v\} + \gamma(v/2 + K\{v\}) = u^i + \gamma v^i.$$

A rigorous proof of the existence and uniqueness of this approach was given by Lin [21]. Uniqueness of the solution requires that the coupling parameter γ has a non-zero imaginary part. A study of the choice of the coupling parameter was given by Amini [22]. However, the differentiation may introduce a strongly singular integral which must be regularized for the equations to be amenable to numerical solution. Several improvements have been introduced, particularly to treat the introduced hypersingularity e.g. [23, 24, 25] or to avoid it completely by applying B&M equations on an interior surface [26].

Previous comparisons between CHIEF and B&M include Amini and Harris [14] and Marburg and Amini [4]. Each method has its own advantages and disadvantages. In particular, the CHIEF is relatively simple but requires the careful search for non-nodal interior points and the treatment of the resulting overdetermined system of equations. The B&M method on the other hand, maintains a square matrix form but requires the careful treatment of the introduced hypersingularity.

The method of Dual Surface Integral Equations (DSIEs) preserves the advantage of the simplicity of CHIEF and specifies the recommended location of the interior points to ensure uniqueness. Thus, the method adds the IHFEs (equation 8) on an appropriately located interior surface S close to Σ constructed at a distance δ along the normal to the surface, to the SIEs (equation 4 or 5) with a purely imaginary factor α to ensure the uniqueness.

$$u/2 - D\{u\} - \alpha\underline{D}\{u\} + S\{v\} + \alpha\underline{S}\{v\} = u^i + \alpha U^i,$$

or $$v/2 + K\{v\} + \alpha\underline{S}\{v\} - M\{u\} - \alpha\underline{D}\{u\} = v^i + \alpha U^i.$$

The resulting system of equations avoids the hypersingularities of B&M while preserving its square matrix form.

Complex coupling parameters related to different integral solutions of the Helmholtz equation were first introduced in Panich [27], Brackage and Werner [28] and Leis [29] and used in B&M [16]. The use of auxiliary interior surfaces is known for example in the source simulation technique (wave superposition, method of fundamental solution) [30], CHI method due to Cunefare *et al.* [26] to avoid the B&M hypersingularity and the off boundary method by Achenbach *et al.* [31]. More recently, the Neumann problem was formulated in terms of single layer potential on both the obstacle and an interior surface thus avoiding the introduction of hypersingularity [32, 33, 34]. In the ICA-RING method [35], the scatterer is hollowed as a shell thus shifting the eigenfrequencies to a higher range.

The proof of the uniqueness of the solution follows the proofs given in [5] and [7]. If we assume that there are two solutions u_1 and u_2, then the difference $u = u_2 - u_1$ satisfies Helmholtz equation in the interior and equal zero on both Σ and S provided that α is imaginary. Taking δ to be less than $\lambda/2$, i.e. $k\delta < \pi$, will ensure that the cavity formed by Σ and S cannot support any resonant modes. Hence u is zero and uniqueness is proved.

5. Application of the DSIE method

Next we present some applications of the DSIE method to selected acoustic scattering and radiation problems. Three objects are considered, a sphere, a cube and a "cat's eye". The finite element models of these objects are shown in Figure 1. Since α and δ are not uniquely defined in the method, a parametric study was also performed.

5.1. Scattering and radiation of a sphere

For the scattering problem we consider a plane wave $U^i = \exp(-ikz)$ incident along the axial direction on a hard and a soft sphere of radius a and center at the origin. For the radiation case, we compute the sound pressure of a pulsating sphere. We closely follow the treatment developed in [18] for axisymmetric problems. The surface integrals are reduced to one along the generating curve C. In this case, the semicircle is discretized in a finite number of segments and the integrals are performed numerically. To ensure accurate results, the length of each segment is less than a quarter wavelength. The numbering of the nodes

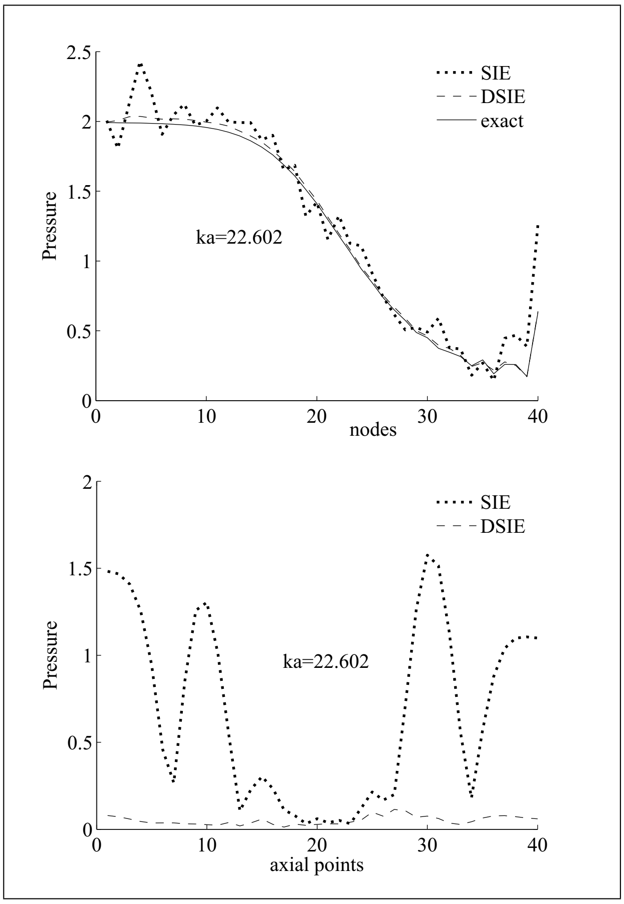

Figure 2. Sound pressure at the surface (top) and at interior points (bottom) of the rigid sphere.

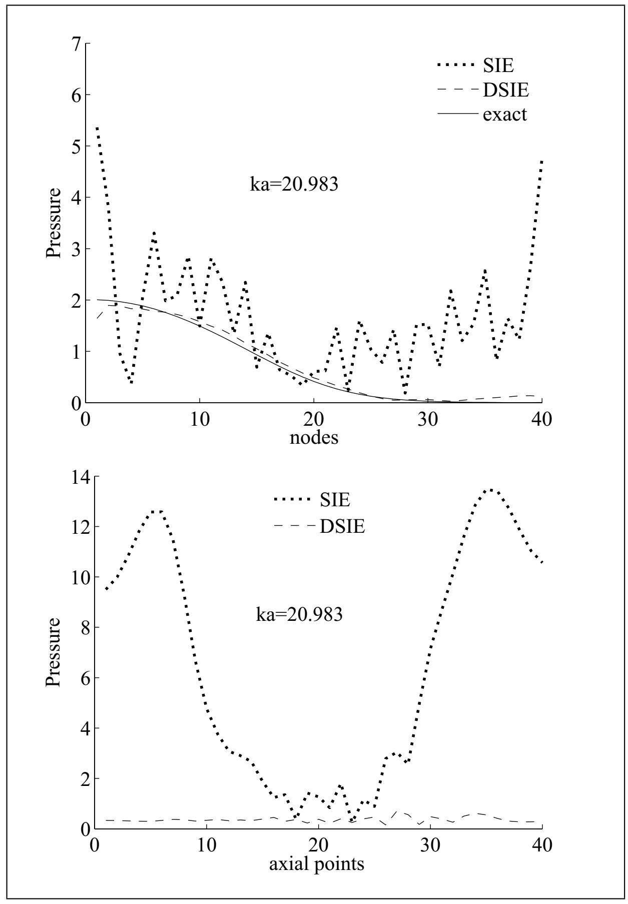

Figure 3. Sound pressure at the surface (top) and at interior points (bottom) of the soft sphere.

starts from the top, i.e. the first node is placed at $z = a$ and the last one at $z = -a$.

For hard scattering ($v = 0$), equations (7) and (8) according to the DSIE method yield

$$u/2 - D\{u\} - \alpha \underline{D}\{u\} = u^i + \alpha U^i. \tag{9}$$

The internal resonances of the sphere are given by the zeros of the Bessel functions and its derivatives. A list of the zeros can be found in [36]. Numerical errors are observed not only at the resonance value but also around it. Employing the DS method, the solution for $ka = 22.602$ compared to the exact solution is shown in Figure 2. This ka which lies in a resonance region exhibits the NU problem as can be seen from the very high axial interior fields. Figure 2 shows the surface and axial interior fields before (SIE) and after corrections (DSIE). Here $k\delta = 2.2$ and $\alpha = -\mathrm{i}$ which satisfy the uniqueness requirements.

For soft scattering ($u = 0$) using Fredholm integral equation of the first kind, equation (6b) coupled to equation (8) yields

$$S\{v\} + \alpha \underline{S}\{v\} = u^i + \alpha U^i. \tag{10}$$

Employing the DS method, the solution for $ka = 20.983$ compared to the exact solution is shown in Figure 3. The figure also shows the axial interior fields before and after corrections. Here $k\delta = 2.1$ and $\alpha = -\mathrm{i}$.

For soft scattering using Fredholm integral equation of the second kind, equation (6a) coupled to equation (8) yields

$$v/2 + K\{v\} + \alpha \underline{S}\{v\} = v^i + \alpha U^i. \tag{11}$$

Employing the DS method, the solution for the resonant value $ka = 22.602$ compared to the exact solution is shown in Figure 4. The figure also shows the axial interior fields before and after corrections. Here $k\delta = 2.2$ and $\alpha = \mathrm{i}k$ which are consistent with the uniqueness as well as the dimension requirements.

For the radiation problem for a specified surface distribution v, the SIE is given by the Fredholm integral equation of the second kind,

$$-u/2 + D\{u\} = S\{v\}. \tag{12}$$

When equation (12) is coupled to equation (8), the result yields

$$-u/2 + D\{u\} + \alpha \underline{D}\{u\} = S\{v\} + \alpha \underline{S}\{v\}. \tag{13}$$

In the numerical example, the pulsating sphere has a normal velocity $v = 1$. Employing the DS method, the solution for $ka = 20.983$ compared to the exact solution is shown in Figure 5. The figure also shows the axial interior fields before and after corrections. Here $k\delta = 2.1$ and

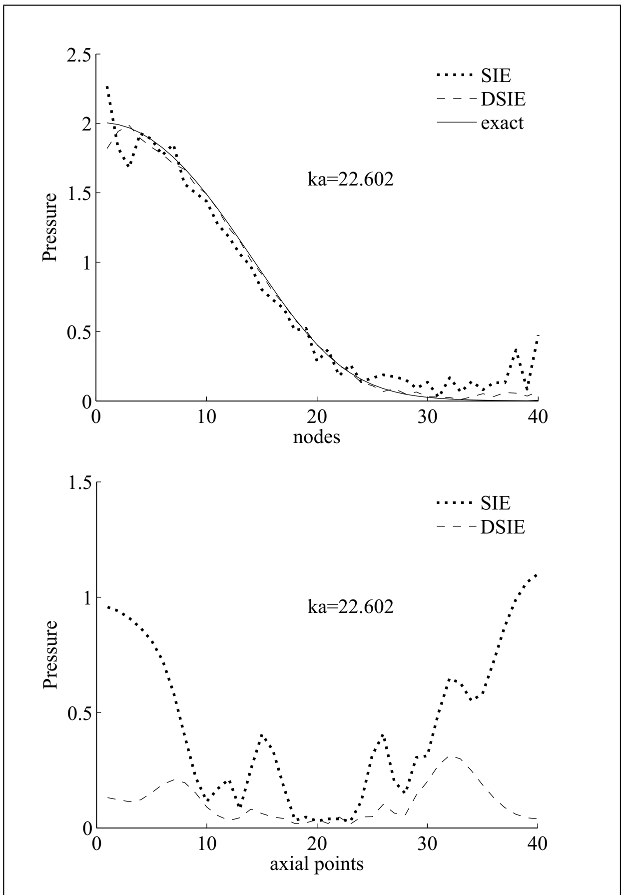

Figure 4. Sound pressure at the surface (top) and at interior points (bottom) of the soft sphere.

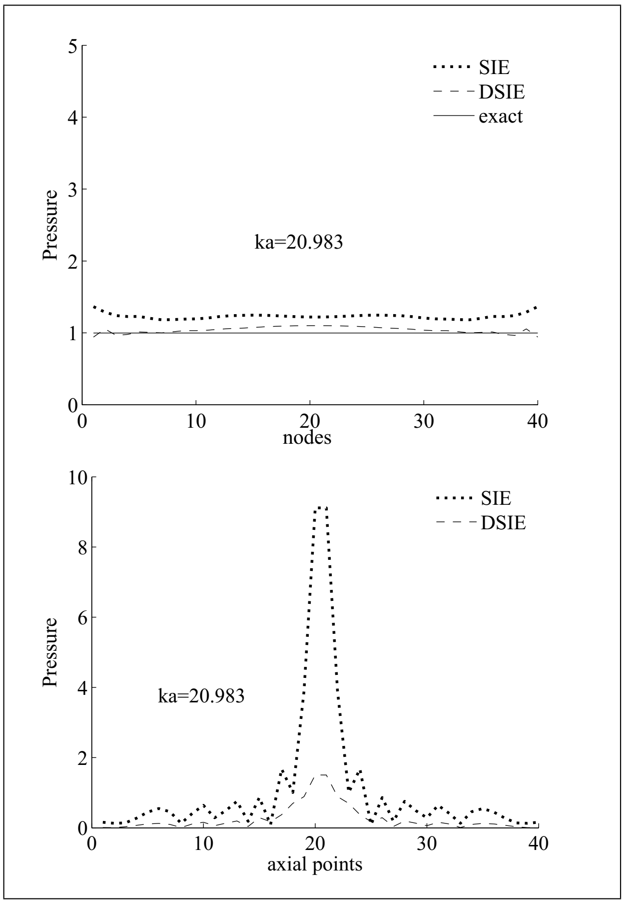

Figure 5. Sound pressure at the surface (top) and at interior points (bottom) of the pulsating sphere.

$\alpha = -\mathrm{i}ka$. This value of the coupling parameter a provides a smaller interior field but smaller values of α also give a good agreement for the surface pressure.

5.2. Scattering and radiation from a cube

We now examine the scattering and radiation of a cube in a certain frequency interval. The three dimensional model of a cube of sides $a = 1$ m is considered. The surface elements are squares of sides 0.1 m which assures at least 6 elements per wavelength up to 500 Hz. It is important to choose an appropriate value for δ that provides accurate results over the whole interval.

For the scattering problem, again a plane wave travelling in the $(0, 0, -1)$ direction is assumed and the cube is rigid. Three different values of δ are tested: a constant value and two frequency-dependent values. The frequency band between 200 Hz and 500 Hz (ka between 3.7 and 9.2) is considered. The coupling parameter α is set to $\alpha = -\mathrm{i}$.

In Figure 6 we observe the pressure level of the forward and backward scattered sound for the analyzed frequency range. In this case, both waves have amplitudes of the same order. Two irregular frequencies can be recognized. The results of the DSIE are compared with the SIE and the B&M method. The two peaks corresponding to the irregular frequencies are eliminated by using the DSIE. For all other frequencies, the results obtained with $\delta = 0.1$ and $\delta = \lambda/8$ are very close to the results of the SIE. The B&M method shows a deviation of about 0.5 dB from the SIE in the whole frequency range. For $\delta = \lambda/4$, there are some oscillations, especially with respect to the backward wave. For the low frequencies, if $\lambda/4$ is chosen, the second surface lies almost in the middle of the cube, which is not an optimum configuration. But for higher frequencies, this value should provide also good results.

The value of the coupling parameter is also tested with this model. We took four values $\alpha = -\mathrm{i}, -2\mathrm{i}, \mathrm{i}$ and $2\mathrm{i}$ for $\delta = 0.1$. Figure 7 shows that there are only very small deviations between the results obtained with the different coupling parameters.

Concerning the radiation problem, we assume that the normal velocity at the surface of the cube is due to a dipole located near the center of the cube (r_d). The sound pressure p_d and sound power W_d of the dipole are given by the expressions

$$p_d(\vec{r}) = h_n^{(1)}(k|\vec{r} - \vec{r}_d|)P_1^0(\cos\gamma),$$
$$W_d = \frac{2\pi}{3\rho c k^2}, \qquad (14)$$

where $h_1^{(1)}(x)$ is the spherical Bessel function of the first kind and first order, $P_1^0(x)$ is the associated Legendre function and γ is the angle between $(r - r_d)$ and the z-axis.

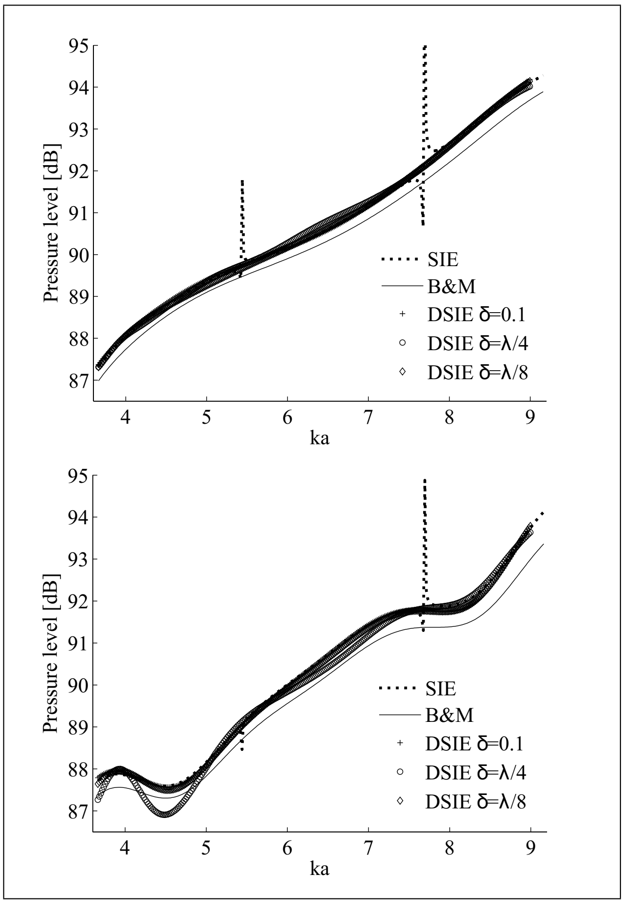

Figure 6. Scattered pressure obtained with the DSIE for different values of δ. Top: forward scattering; bottom: backward scattering.

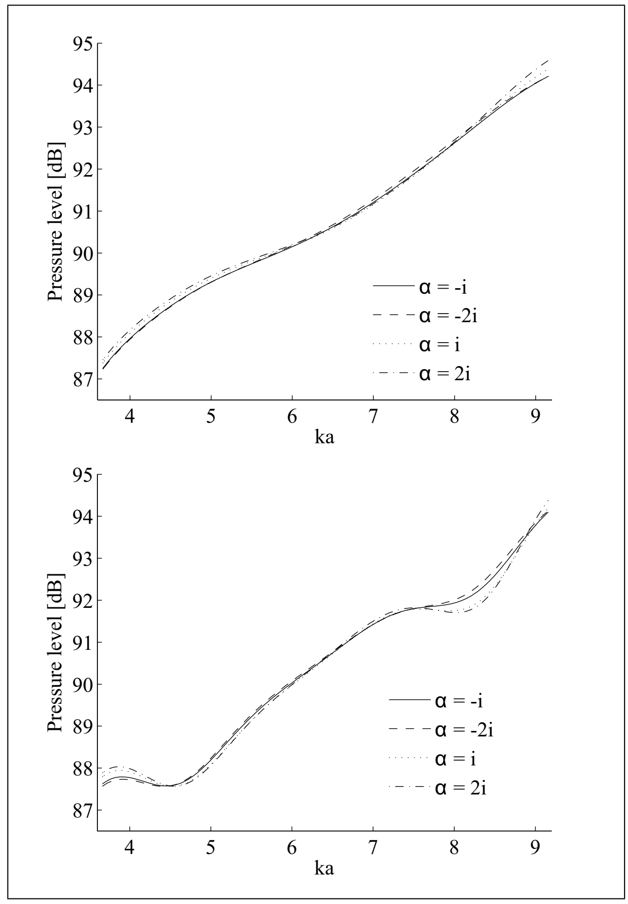

Figure 7. Scattered pressure obtained with the DSIE for different values of α. Top: forward scattering; bottom: backward scattering.

We evaluate the sound power produced by the vibrating surface of the cube and compare it to the sound power of the dipole given in (14). For the DSIE, the distance of the second surface is set to $\delta = \lambda/8$ and the coupling parameter to $\alpha = -\mathrm{i}$. The result is compared to the SIE and the B&M method as well. Figure 8 shows the good performance of the DSIE, whose result differs minimally from the exact solution.

5.3. Scattering and radiation of a cat's eye

The DSIE approach is especially useful at high frequencies where the density of interior resonances increases and makes the application of the CHIEF method very difficult. In this example we move the frequency range towards the high frequencies. The number of elements is also increased to ensure accurate results, but a direct solver could still be used. For even higher frequencies and finer discretization, iterative solvers need to be introduced (see for example [37, 38, 39, 40]). The distance between the original and second surface was $\delta = \lambda/8$ and the coupling parameter was $\alpha = -\mathrm{i}$.

For the scattering problem, an incident plane wave in the direction $(0, 0, -1)$ and a rigid surface are considered. The wave strikes one of the plane surfaces perpendicularly. The studied frequency range goes from $ka = 10$ to $ka = 20$.

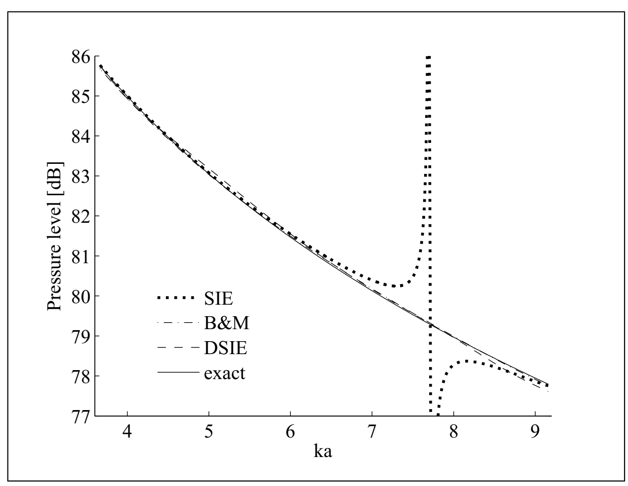

Figure 8. Sound power of the dipole obtained with the cube.

Figure 9 shows the spectral forward scattering and the spatial distribution of the scattered wave at one of the irregular frequencies.

The high density of interior resonances is evident. To be able to detect them, the resolution along the ka-axis is very fine, smaller than 0.02. The DSIE eliminates all the irregular frequencies and provides a smooth curve. The difference between the DSIE and the B&M does not exceed

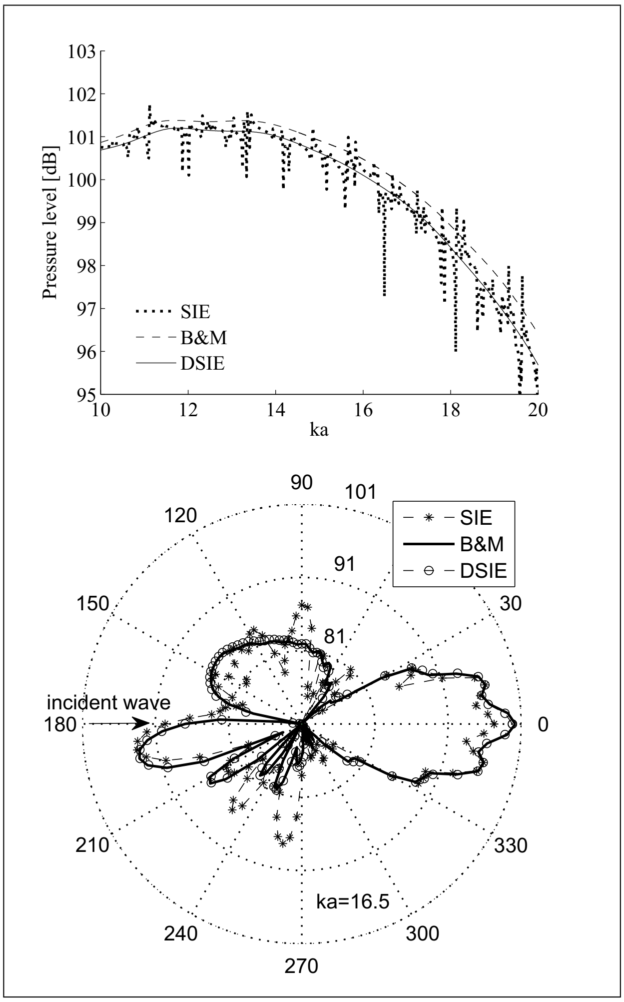

Figure 9. Scattering from a rigid cat's eye.

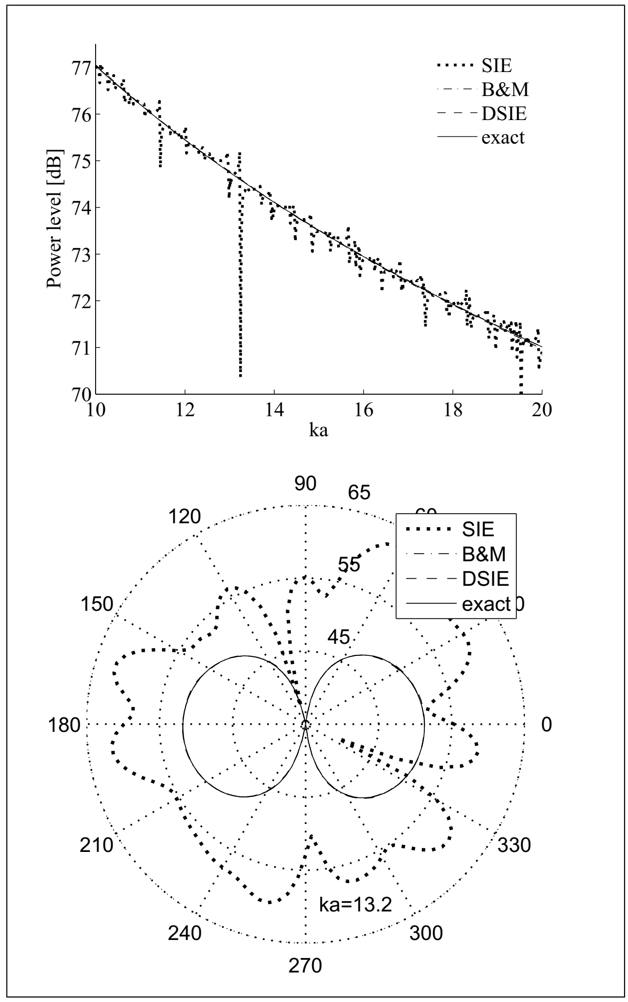

Figure 10. Top: Sound power of the dipole obtained with the cat's eye, bottom: directivity pattern of the sound pressure level.

0.5 dB in the whole frequency range. The polar plot shows the scattered pressure around the object at one irregular frequency ($ka = 16.5$). The DSIE and B&M curves agree very well while the SIE result is close in the forward and backward lobes but differ at the side lobes.

For the radiation problem, again a dipole test is performed. The spectral sound power and the directivity of the sound pressure at one irregular frequency are illustrated in Figure 10. The B&M and the DSIE methods provide very good results. The typical eight-pattern of the dipole is very well reproduced at $ka = 13.2$ whereas the SIE gives completely wrong results. The deviations from the analytical power curve are minor.

6. Discussion

The problem of NU of the solution of acoustic problems via BIEs is considered. We stress on the application of the DSIE method to exterior radiation and scattering acoustic problems. The DSIEs maintain the simplicity of the original equations and provide a unique solution for all frequencies as long as the coupling constant α is imaginary and the interior auxiliary surface is located at a distance δ which is less than $\lambda/2$. The method avoids the introduction of hypersingular integrals as in the B&M method. The method also solves the problem of how to choose the interior points in CHIEF besides avoiding the introduction of an overdetermined system of equations.

We note that in [6, p.1400], it was pointed out that it is not necessary for uniqueness of solution to keep the inner surface at a fixed distance from Σ as long as δ is less than $\lambda/2$ and large enough that numerical errors in computing the integrands are small. In practice, according to [6], it is not even necessary to include the inner surface for every observation point or to make the inner surface a continuous one. This means that one can avoid any difficulty in defining the inner surface for scatterers with complicated shapes.

We demonstrated the validity of the method via application to scattering and radiation by different objects. For the axisymetric sphere, the problem was investigated at certain frequencies. We chose $ka = 22.602$ which is a high frequency located in a resonance region for the hard case. Although the value is not a resonance one, the results exhibit the common features of NU. Soft scattering was treated at this ka via Fredholm integral equations of the second kind. For the soft case using Fredholm integral equation of the first kind as well for the radiation problem we took $ka = 20.983$. The effect of internal resonance is demonstrated by the high value of the axial internal fields in these cases. The significant reduction in these fields illustrates the success of the method. For the 3D cube and the cat's eye, the acoustic responses were investigated for a broad frequency range. It was found that a value of $\delta = \lambda/8$ and $\alpha = -\mathrm{i}$ provided very good results for the entire frequency range. At low frequencies, care must be taken to avoid that the second surface lie too far from the original surface, since the wave length may be large. For these two cases, the interior field was not investigated. Other values of δ and α may minimize the interior field, but for exterior problems, that may be of minor importance.

It is important to note that the proof of uniqueness sets the necessary conditions, namely, a complex value for α and $\delta < \lambda/2$ but do not provide unique values. For the soft scattering using Fredholm integral equation of the first kind as well as for hard scattering, α was simply taken equals to $-\mathrm{i}$. For soft scattering using Fredholm integral equation of the second kind $\alpha = \mathrm{i}k$ was chosen to make the dimension of the two added terms equal. However for the radiation from a sphere $\alpha = -\mathrm{i}ka$ was required to reduce the interior field. Further studies, mathematical or numerical, are needed to find the most appropriate values for α and δ within the limits given by the uniqueness theory. Similar previous studies for the coupling parameters were conducted by Amini [22] and Kirkup [41].

Acknowledgement

The first author sincerely acknowledges the financial support of the Alexander von Humboldt foundation.

References

[1] A. J. Burton: The solution of Helmholtz equation in exterior domains using integral equation. NPL Rept. NAC 30, Teddington, England, Jan. 1973.

[2] W. Benthien, A. Schenck: Nonexistence and nonuniqueness problems associated with integral equation methods in acoustics. Computers & Structures **65** (1997) 295–305.

[3] S. Marburg, B. Nolte (eds.): Computational acoustics of noise propagation in fluids, finite and boundary element methods. Springer, Berlin, 2008.

[4] S. Marburg, S. Amini: Cat's eye radiation with boundary elements: comparative study on treatment of irregular frequencies. JCA **13** (2005) 21–45.

[5] M. B. Woodworth, A. D. Yaghjian: Derivation, application and conjugate gradient solution of dual-surface integral equations for three-dimensional, multiwavelength perfect conductors. PIERS **5** (1991) 103–130.

[6] M. B. Woodworth, A. D. Yaghjian: Multiwavelength three-dimensional scattering wih dual-surface integral equations. J. Opt. Soc. Amer. A **11** (1994) 1399–1413.

[7] A. Zaporozhets, M. F. Levy: Current marching technique for electromagnetic scattering computations. IEEE Tr. AP- **47** (1999) 1016–1024.

[8] V. V. S. Prakash, R. Mittra: Dual surface combined field integral equation for three-dimensional scattering. Microwave Opt. Tech. Letts. **29** (2001) 293–296.

[9] R. A. Shore, A. D. Yaghjian: Dual-surface integral equations in electromagnetic scattering. IEEE Tr. AP- **53** (2005) 1706–1709.

[10] A. F. Seybert, B. Soenarko, F. J. Rizzo, D. J. Shippy: An advanced computational method for radiation and scattering of acoustic waves in three dimensions. JASA **77** (1985) 362–368.

[11] C. A. Klein, R. Mittra: An application of the condition number concept in the solution of scattering problems in the presence of interior resonant frequencies. IEEE Trans. AP- **23** (1975) 431–435.

[12] C. A. Klein, R. Mittra: Stability of matrix equations arising in electromagnetics. IEEE Trans. AP- **21** (1973) 902–905.

[13] F. X. Canning: Singular value decomposition of integral equations of electromagnetic and applications to the cavity resonance problem. IEEE Trans. AP- **33** (1989) 1153–1163.

[14] S. Amini, P. J. Harris: A comparison between various boundary integral forms of the exterior acoustic problem. Comp. Meth. Appl. Mech. Eng. **84** (1990) 59–75.

[15] H. A. Schenck: Improved integral equation formulation for acoustic radiation problems. JASA **44** (1968) 41–58.

[16] A. J. Burton, G. F. Miller: The application of integral equation methods to numerical solution of some exterior boundary-value problems. Proc. R. Soc. London, Ser. A **323** (1971) 201–210.

[17] A. F. Seybert, T. K. Regarajan: The use of CHIEF to obtain unique solutions for acoustic radiation using integral equations. JASA **81** (1987) 1299–1306.

[18] A. F. Seybert, B. Soenarko, F. J. Rizzo, D. J. Shippy: A special integral equation formulation for acoustic radiation and scattering for axisymmetric bodies and boundary conditions. JASA **80** (1986) 1241–1247.

[19] D. J. Segalman, D. W. Lobitz: A method to overcome computational difficulties in the exterior acoustic problem. JASA **91** (1992) 1855–1861.

[20] A. Mohsen, M. Ochmann: Numerical experiments using chief to treat the nonuniqueness in solving acoustic axisymmetric exterior problems via boundary integral equations. J. Adv. Res. **1** (2010) 227–232.

[21] T. C. Lin: A proof for the Burton and Miller integral equation approach for the Helmholtz equation. J. Math. Anal. Appl. **103** (1984) 565–574.

[22] S. Amini: On the choice of the coupling parameter in boundary integral formulations of the exterior acoustic problem. Applicable Anal. **35** (1990) 75–92.

[23] Z. Y. Yan, K. C. Hung, H. Zheng: Solving the hypersingular boundary integral equation in three-dimensional acoustics using a regularization relationship. JASA **113** (2003) 2674–2683.

[24] A. Osetrov, M. Ochmann: A fast and stable numerical solution for acoustic boundary element method equations combined with the Burton and Miller method for huge models consisting of constant elements. JCA **13** (2005) 1–20.

[25] S. Li, Q. Huang: An improved form of the hypersingular boundary integral equation for exterior acoustic problems. Eng. Anal. With Boundary Elements **34** (2010) 189–195.

[26] K. A. Cunefare, G. Koopmann: A boundary element method for acoustic radiation valid for all wavenumbers. JASA **85** (1988) 39–48.

[27] O. I. Panich: On the question of the solvability of the exterior boundary value problem for the wave equation and Maxwell's equation. Russ. Math. Surv. **20** (1965) 221–226.

[28] H. Brakhage, P. Werner: Über das Dirichlet'sche Außenraumproblem für die Helmholtz'sche Schwingungsgleichung. Archiv der Math. **16** (1965) 325–329.

[29] R. Leis: Zur Dirichletschen Randwertaufgabe des Außenraumes der Schwingungsgleichung. Math. Z. **90** (1965) 205–211.

[30] F. P. Mechel (ed.): Formulas of acoustics. Springer, Berlin, 2002. Section O.4.

[31] J. D. Achenbach, J. D. Kechter, Y.-L. Xu: Off boundary approach to the boundary element method. Comp. Meth. Appl. Mech. Eng. **70** (1988) 191–201.

[32] P. A. Kruttskii: A new approach to reduction of the Neumann problem in acoustic scattering to a non-hypersingular integral equation. IMA J. Appl. Math. **64** (2000) 259–269.

[33] S. A. Yang: A boundary integral equation method using auxiliary interior surface approach for acoustic radiation and scattering in two dimensions. JASA **112** (2002) 1307–1317.

[34] S. A. Yang: An integral equation approach to three-dimensional acoustic radiation and scattering problems. JASA **116** (2004) 1372–1380.

[35] K. Hirosawa, T. Ishizuka, K. Fujiwara: A simple method avoiding non-uniqueness in the boundary element method for acoustic scattering problem. JASA **125** (2009) 2938–2946.

[36] M. Abramowitz, I. A. Stegun (eds.): Handbook of mathematical functions. Dover Pub. Inc., N.Y., 1968. 467-468.

[37] S. Makarov, M. Ochmann: An iterative solver of the Helmholtz integral equation for high-frequency acoustic scattering. JASA **103** (1998) 742–750.

[38] S. Makarov, R. Ludwig, R. Lemdiasov, M. Ochmann: An iterative solution for magnetic field integral equation. Electromagnetics **22** (2002) 461–471.

[39] S. Makarov, R. Vedantham: Performance of the generalized minimum residual (GMRES) iterative solution for the magnetic field integral equation. Radio Science **37** 1072. doi:10.1029/2000RS002588.

[40] M. Ochmann, A. Homm, S. Makarov, S. Semenov: An iterative GMRES-based boundary element solver for acoustic scattering. Engineering Analysis with Boundary Elements **27** (2003) 717–725.

[41] S. M. Kirkup: The influence of the weighting parameter on the improved boundary element solution of the exterior Helmholtz equation. Wave Motion **15** (1992) 93–101.